ON p-HARMONIC MORPHISMS

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Abstract. In this paper, we study the characterisation of p-harmonic morphisms between Riemannian manifolds, in the spirit of Fuglede-Ishihara. After a result comparing (2-)harmonic morphisms and p-harmonic morphisms ($p \neq 2$), we establish that p-harmonic morphisms are precisely horizontally weakly conformal p-harmonic maps.

1. INTRODUCTION

A particularly interesting subclass of harmonic maps is the harmonic morphisms, i.e. maps between Riemannian manifolds which pull back local harmonic functions to harmonic functions. Though their history goes back as far as Jacobi in 1848 (cf. [15]), the real starting point, as far as differential geometry is concerned, was their characterisation published independently by Fuglede [10] and Ishihara [14] in the late 1970's. This theorem shows that harmonic morphisms encompass both analytical and geometrical features and therefore their study yields many results on the relationship between the differentiable structure of a manifold and its geometry.

Viewing harmonic maps as critical points of the energy integral, their most natural generalisation is the notion of p-harmonic maps, i.e. critical points of the p-energy integral, as stated in Definition 5. With this in mind, it is difficult to resist the temptation of defining p-harmonic morphisms as maps between Riemannian manifolds which pull back local p-harmonic functions to p-harmonic functions. For $p = 2$, we obviously recover harmonic morphisms. After a section recalling the basics of harmonic morphisms and another dedicated to p-harmonic maps, we show, in Proposition 3, that, for $p \neq 2$, a harmonic morphism is also a p-harmonic morphism if and only if it is horizontally homothetic.

In the last and main section, we obtain a Fuglede-Ishihara type characterisation for p-harmonic morphisms.

Finally, we would like to point out that another natural generalisation of harmonic morphisms, is to consider semi-Riemannian metrics. This is the path followed by Fuglede in [11]), where the results obtained mirror those of [10, 14].

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2. BASIC FACTS ON HARMONIC MORPHISMS

Definition 1. A map $\phi : (M, g) \to (N, h)$ is a (2)-*harmonic morphism* if for any harmonic function f defined on an open set $V \subseteq N$, such that $\phi^{-1}(V) \neq \emptyset$, then the composition $f \circ \phi : \phi^{-1}(V) \to \mathbb{R}$ is a harmonic function.

Remark 1.
1. It is clear that the notion of harmonic morphism is local. Indeed the assignment to every non-empty open subset U in M of the set of all harmonic morphisms from U to N defines a sheaf (cf. [10]).
2. Harmonic morphisms can be characterised as mapping Brownian motions (i.e. random paths on a manifold) to Brownian motions up to a change of time scale (cf. [4, 5, 16]).

Definition 2. Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. We decompose the tangent space at a point $x \in M$ as

$$T_x M = T_x^H M \oplus T_x^V M,$$

where $T_x^V M = \ker d\phi_x$ is called the *vertical space* and $T_x^H M = (\ker d\phi_x)^\perp$ the *horizontal space*.

Let C_ϕ be the set of *critical points*, i.e. $C_\phi = \{x \in M \mid \operatorname{rank}(d\phi_x) \text{ is not maximal}\}$, we write $\widetilde{M} = M \setminus C_\phi$ for the set of *regular points*.

The *vertical distribution* is defined as $\mathcal{V} = \left\{T_x^V M \mid x \in \widetilde{M}\right\}$ and the *horizontal distribution* by $\mathcal{H} = \left\{T_x^H M \mid x \in \widetilde{M}\right\}$.

Definition 3. A map $\phi : (M^m, g) \to (N^n, h)$ is called *horizontally weakly conformal* if for all non-critical points x (i.e. $x \in \widetilde{M}$), $d\phi_x \Big|_{T_x^H M^m}$ is conformal and surjective.

Then there exists a function $\lambda : \widetilde{M} \to \mathbb{R}^+$ such that

$$\lambda^2 g_x(X, Y) = (h_\phi)_x \left(d\phi_x(X), d\phi_x(Y)\right) \qquad \forall X, Y \in T_x^H M^m, \forall x \in \widetilde{M}. \tag{1}$$

The function λ is extended continuously to the whole of M by setting $\lambda|_{C_\phi} \equiv 0$ and called the *dilation* of ϕ.

Note that at any point $x \in M$, $\operatorname{rank} d\phi_x$ is either n or 0 and that the map ϕ is constant if and only if $\lambda \equiv 0$.

Definition 4. A non-constant horizontally weakly conformal map $\phi : (M^m, g) \to (N^n, h)$ of dilation λ, such that the gradient of λ^2 is vertical, i.e. $\operatorname{grad}_{\mathcal{H}} \lambda^2 \equiv 0$, is called *horizontally homothetic*.

Remark 2.
1. It follows from the definition of horizontally weakly conformal that the *energy density* $e(\phi)$, defined by $e(\phi) = \frac{1}{2} \mid d\phi \mid^2$, is equal to $\frac{n}{2}\lambda^2$ and therefore λ^2 is a smooth function on the whole of M; however, in general, λ is only continuous at critical points.
2. Because the map ϕ is a submersion on the set $\widetilde{M}$, if in Definition 3 we have $m < n$, then ϕ must be a constant map.

The following characterisation (proved independently by Fuglede and Ishihara) shows that harmonic morphisms act as a link between the analytical and the geometrical properties of Riemannian manifolds.

Theorem 1. [10, 14] *A smooth map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally weakly conformal harmonic map.*

In fact harmonic morphisms can be seen as mappings preserving the Laplacian (up to a factor):

Proposition 1. [10] *The map $\phi : (M, g) \to (N, h)$ is a harmonic morphism with dilation λ if and only if*

$$\Delta^M(f \circ \phi) = \lambda^2 \Delta^N(f) \circ \phi$$

for any $f \in C^2(N, \mathbb{R})$.

A pleasing consequence of Theorem 1 (observed in [8]) is that harmonic morphisms pull back (local) harmonic *maps* to (local) harmonic *maps* as well.

3. p-HARMONIC MAPS

Definition 5. Let (M^m, g) and (N^n, h) be two Riemannian manifolds. Let $p \in]2, +\infty[$, a map $\phi : (M^m, g) \to (N^n, h)$ is called a *p-harmonic map* if ϕ is a critical point of the integral

$$E_p(\phi, \Omega) = \frac{1}{p} \int_\Omega \mid d\phi \mid^p \ v_g$$

for all compact subset $\Omega \subseteq M$.

In contrast with the $p = 2$ case, infinite dimensional Morse Theory can be applied to the functional $E_p(., \Omega)$ as it is known to satisfy Condition (C) of Palais-Smale if $p > m$ (cf. [20]).
The Euler-Lagrange equation associated with E_p is (cf. [3])

$$\tau_p(\phi) = \mid d\phi \mid^{p-4} \left[\mid d\phi \mid^2 \tau_2(\phi) + \frac{(p-2)}{2} d\phi \, \mathrm{grad}_M(\mid d\phi \mid^2) \right] \tag{2}$$

where $\tau_2(\phi)$ is the tension field $\tau(\phi)$ for harmonic maps.
This operator is a system of quasi-linear elliptic differential equations (only for $p = 2$ do they become semi-linear) which can be put in divergence form (see [7])

$$\mathrm{div} \left(|d\phi|^{p-2} \, d\phi \right) = 0$$

and thus satisfies a maximal principle (cf. [12]).
Many regularity results have been obtained: Takeuchi shows in [19] that a weakly p-harmonic map from an open subset of $\mathbb{R}^m$ to the unit sphere $\mathbb{S}^n$ is locally Hölder continuous almost everywhere, while Hardt and Lin [13] and Luckhaus [17] have proved that minimising p-harmonic maps are locally Hölder continuous on the complement of a compact set.

An existence theorem extending that of Eells-Sampson [9] was showed by Duzaar and Fuchs [6] and a Liouville theorem for p-harmonic maps has been obtained by Takeuchi in [18].

4. A COMPARISON STUDY

Definition 6. A map $\phi : (M^m, g) \to (N^n, h)$ is called a *p-harmonic morphism* (or *morphism of p-harmonic functions*) if it pulls back (local) p-harmonic functions on N to (local) p-harmonic functions on M, i.e.

$$\tau_p(f) = 0 \quad \Rightarrow \quad \tau_p(f \circ \phi) = 0 \quad \forall f \in C^2(V, \mathbb{R}) \quad V \overset{\text{open}}{\subseteq} N.$$

Proposition 2. *Let (N^n, h) be a Riemannian manifold. Then for any point $q \in N$ and any set of constants $\{C_{\alpha\beta}\}$ $(C_{\alpha\beta} = C_{\beta\alpha})$, $\{C_\alpha\}$ $\alpha, \beta \in \{1, \dots, n\}$, satisfying*

$$\left(\sum_\alpha C_\alpha^2\right)\left(\sum_\alpha C_{\alpha\alpha}\right) + (p-2)\sum_{\beta,\delta=1}^{n} C_\beta C_\delta C_{\beta\delta} = 0, \tag{3}$$

there exists a p-harmonic function f defined on a neighbourhood $U \subseteq N$ of q such that in a system of normal local coordinates (y^α) centred on q

$$\frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}(q) = C_{\alpha\beta}$$
$$\frac{\partial f}{\partial y^\alpha}(q) = C_\alpha$$

Proof. This is an application of [1, Proposition 2.4], which basically states that, for quasi-linear elliptic operators, infinitesimal solvability, i.e. Condition (3), implies local solvability. □

Remark 3. For the case $p = 2$, Proposition 2 was proved by Ishihara in [14], while Fuglede in [11] extended this result to the case of semi-Riemannian manifolds (still for the case $p = 2$).

Throughout the rest of this paper, for a given point $x \in M^m$, we shall equip (M^m, g) with a system of normal coordinates $(x^i)_{i=1,\dots,m}$ around x and (N^n, h) with normal coordinates $(y^\alpha)_{\alpha=1,\dots,n}$ centred on $\phi(x)$.

Proposition 3. *Let $\phi : (M, g) \to (N, h)$ be a non-constant 2-harmonic morphism between two Riemannian manifolds and let $p \in]2, +\infty[$. Then ϕ is a p-harmonic morphism if and only if it is horizontally homothetic.*

Proof. Let $\phi : M \to N$ be a non-constant 2-harmonic morphism with dilation function λ and let f be a local p-harmonic function on (N^n, h). Applying the chain rule

$$\tau_p(f \circ \phi) = | d(f \circ \phi) |_g^{p-4} \left[| d(f \circ \phi) |_g^2 \, \tau_2(f \circ \phi) + \frac{(p-2)}{2} d(f \circ \phi) \operatorname{grad}_M(| d(f \circ \phi) |_g^2) \right]$$

$$= | d(f \circ \phi) |_g^{p-4} \left[| d(f \circ \phi) |_g^2 \left(df \circ \tau_2(\phi) + \lambda^{2N} \Delta f \right) + \frac{(p-2)}{2} d(f \circ \phi) \operatorname{grad}_M(| d(f \circ \phi) |_g^2) \right]$$

We first look at the term $\mid d(f\circ\phi)\mid_g^2$:

$$\mid d(f\circ\phi)\mid_g^2 = \partial_i(f\circ\phi)\partial_j(f\circ\phi)g^{ij} = f_\alpha\phi_i^\alpha f_\beta\phi_j^\beta g^{ij} = f_\alpha f_\beta\lambda^2 h^{\alpha\beta} = \lambda^2 \mid df\mid_h^2 \circ\phi.$$

For any function k on N we have:

$$d(f\circ\phi)\,\mathrm{grad}_M\,(k\circ\phi) = \lambda^2 df\,\mathrm{grad}_N(k)\circ\phi$$

since, in normal local coordinates,

$$d(f\circ\phi)\,\mathrm{grad}_M\,(k\circ\phi) = \sum_{\alpha,\beta,i,j} f_\alpha\phi_i^\alpha g^{ij}k_\beta\phi_j^\beta = \lambda^2\sum_{\alpha,\beta} f_\alpha h^{\alpha\beta}k_\beta = \lambda^2 df\,\mathrm{grad}_N(k)\circ\phi.$$

Now we go back to $\tau_p(f\circ\phi)$ which, assuming that ϕ to be a p-harmonic morphism, we expand using the remarks made above:

$$\begin{aligned}
\tau_p(f\circ\phi) =& \mid d(f\circ\phi)\mid_g^{p-4}\Big[\mid d(f\circ\phi)\mid_g^2\left(df\circ\tau_2(\phi) + \lambda^{2\,N}\Delta f\right)\\
&+\frac{(p-2)}{2}d(f\circ\phi)\,\mathrm{grad}_M(\mid d(f\circ\phi)\mid_g^2)\Big]\\
=& \mid d(f\circ\phi)\mid_g^{p-4}\Big[\mid d(f\circ\phi)\mid_g^2\left(\lambda^{2\,N}\Delta f\right)\\
&+\frac{(p-2)}{2}d(f\circ\phi)\,\mathrm{grad}_M(\lambda^2\mid df\mid_h^2)\Big] \quad \text{since } \phi \text{ is harmonic}\\
=& \mid d(f\circ\phi)\mid_g^{p-4}\Big[\mid d(f\circ\phi)\mid_g^2\left(\lambda^{2\,N}\Delta f\right)\\
&+\frac{(p-2)}{2}d(f\circ\phi)\left(\lambda^2\,\mathrm{grad}_M(\mid df\mid_h^2) + \mid df\mid_h^2\,\mathrm{grad}_M(\lambda^2)\right)\Big]\\
=& \lambda^{p-4}\mid df\mid_h^{p-4}\Big[\lambda^2\mid df\mid_h^2\lambda^{2\,N}\Delta f\\
&+\frac{(p-2)}{2}\left[\lambda^4 df\left(\mathrm{grad}_N(\mid d(f)\mid_h^2)\right) + \mid df\mid_h^2\, df\circ d\phi\left(\mathrm{grad}_M(\lambda^2)\right)\right]\Big]\\
=& \lambda^p\mid df\mid_h^{p-4}\Big[\mid df\mid_h^{2\;N}\Delta f + \frac{(p-2)}{2}df\,\mathrm{grad}_N(\mid d(f)\mid_h^2)\Big]\\
&+\frac{(p-2)}{2}\lambda^{p-4}\mid df\mid_h^{p-2}\,df\circ d\phi\,\mathrm{grad}_M(\lambda^2)\\
=& \lambda^p\tau_p(f) + \frac{(p-2)}{2}\lambda^{p-4}\mid df\mid_h^{p-2}\,df\circ d\phi\,\mathrm{grad}_M(\lambda^2)\\
=& \frac{(p-2)}{2}\lambda^{p-4}\mid df\mid_h^{p-2}\,df\circ d\phi\,\mathrm{grad}_M(\lambda^2)\\
=& 0.
\end{aligned}$$

Since we assume that $p\neq 2$ this implies

$$df\circ d\phi\left(\mathrm{grad}_M(\lambda^2)\right) = 0. \tag{4}$$

According to Proposition 2, we can choose the p-harmonic function f to be such that

$$\frac{\partial f}{\partial y^\alpha}(q) = \delta_{\alpha\alpha_0},$$

for a fixed α_0, showing that (4) implies

$$d\phi\left(\mathrm{grad}_M \lambda^2\right) \equiv 0$$

i.e. λ is vertical, thus ϕ is horizontally homothetic.
By reversing this argument we prove the converse. □

Remark 4. As was pointed out by P. Baird, Proposition 3 could also be proved using the Stress-Energy Tensor of [2].

Corollary 1. *If ϕ is a 2-harmonic morphism and a p-harmonic morphism for* some $p \neq 2$ *then ϕ is a p-harmonic morphism for all $p \in \mathbb{N}$.*

5. p-HARMONIC MORPHISMS

Let us choose $p \in]2, +\infty[$ and let (M^m, g) and (N^n, h) be two Riemannian manifolds.

Proposition 4. *Let $\phi : (M^m, g) \to (N^n, h)$ be a horizontally weakly conformal p-harmonic map, then ϕ is a p-harmonic morphism.*

Proof. Let $f : (N, h) \to \mathbb{R}$ be a function.
We have the following formula (valid away from points at which $d\phi \equiv 0$) for $\tau_p(f \circ \phi)$ (due to Takeuchi [19]):

(5)
$$\tau_p(f\circ\phi) = \theta^{p-2} df\left(\tau_p(\phi)\right) + \mid d\phi\mid^{p-2}\left\{\theta^{p-2}\,\mathrm{trace}(\nabla df)\,(d\phi, d\phi) + d(f\circ\phi)\,\mathrm{grad}\,\theta^{p-2}\right\},$$

where $\theta = \frac{|d(f\circ\phi)|}{|d\phi|}$.
Using this formula we can see that any p-harmonic map ϕ which is horizontally weakly conformal will pull back (local) p-harmonic functions to (local) p-harmonic functions.
Indeed for a horizontally weakly conformal map ϕ with dilation λ:

$$\mid d(f\circ\phi)\mid^2 = \lambda^2 \mid df \mid^2, \tag{6a}$$

$$\mid d\phi \mid^2 = n\lambda^2, \tag{6b}$$

$$\mathrm{trace}(\nabla df)\,(d\phi, d\phi) = \lambda^2 \Delta^N f, \tag{6c}$$

$$d(f\circ\phi)\,\mathrm{grad}_M(g\circ\phi) = \lambda^2 df\,\mathrm{grad}_N g \quad \forall g \in C^1(N, \mathbb{R}), \tag{6d}$$

and these relations imply that:

$$\begin{aligned}\tau_p(f\circ\phi) = &\ \frac{\mid df\mid^{p-2}}{n^{\frac{p-2}{2}}} df\left(\tau_p(\phi)\right) + \lambda^p \tau_p(f)\\ =&\lambda^p\tau_p(f) = 0.\end{aligned} \tag{7}$$

Remark that because each of the Relations (6) is still valid with f replaced by a *map* $\psi : N \to P$, we have that for a horizontally weakly conformal map $\phi : M \to N$:

$$\tau_p(\psi \circ \phi) = \frac{\mid d\psi \mid^{p-2}}{n^{\frac{p-2}{2}}} d\psi\left(\tau_p(\phi)\right) + \lambda^p \tau_p(\psi)$$

which shows that a horizontally weakly conformal p-harmonic map pulls back p-harmonic maps to p-harmonic maps. □

Proposition 5. *Let $\phi : (M^m, g) \to (N^n, h)$ be a p-harmonic map and a p-harmonic morphism then ϕ is horizontally weakly conformal.*

Proof. Let $q \in M$ be a point such that $d\phi(q)$ does not vanish, and let us choose normal local coordinates on (M, g) centred around q and centred around $\phi(q)$ on (N, h).
We put

$$\frac{\mid d\phi^\alpha \mid^2}{\mid d\phi \mid^2} = K_\alpha,$$
$$\frac{< d\phi^\alpha, d\phi^\beta >}{\mid d\phi \mid^2} = K_{\alpha\beta},$$

all the differentials being at the point q.
Of course $K_{\alpha\alpha} = K_\alpha$ and if $d\phi^\alpha \equiv 0$ for a specific α, then $K_\alpha = K_{\alpha\beta} = 0 \quad \forall \beta$.

Since ϕ is a p-harmonic map, using Equation (5) we have, at the point q (and all subsequent formulae are pointwise), for any function f:

$$\tau_p(f \circ \phi) = \mid d\phi \mid^2 \mid d(f \circ \phi) \mid^{p-4} \left\{ \frac{\mid d(f\circ\phi) \mid^2}{\mid d\phi \mid^2} \operatorname{trace}(\nabla df)\left(d\phi, d\phi\right) \right.$$
$$\left. + \frac{p-2}{2} d(f \circ \phi) \operatorname{grad} \frac{\mid d(f\circ\phi) \mid^2}{\mid d\phi \mid^2} \right\}.$$

In local normal coordinates we have:

$$\begin{aligned}\operatorname{trace}(\nabla df)\left(d\phi, d\phi\right) &= \sum_{\alpha,\beta,i} f_{\alpha\beta} \phi_i^\alpha \phi_i^\beta \\ &= \sum_\alpha f_{\alpha\alpha} \mid d\phi^\alpha \mid^2 + \sum_{\alpha\neq\beta} f_{\alpha\beta} < d\phi^\alpha, d\phi^\beta >,\end{aligned}$$

therefore

$$\frac{1}{\mid d\phi \mid^2} \operatorname{trace}(\nabla df)\left(d\phi, d\phi\right) = \sum_\alpha f_{\alpha\alpha} K_\alpha + \sum_{\alpha\neq\beta} f_{\alpha\beta} K_{\alpha\beta}.$$

Similarly it is easy to see that:

$$\frac{\mid d(f\circ\phi) \mid^2}{\mid d\phi \mid^2} = \sum_\beta f_\beta^2 K_\beta + \sum_{\gamma\neq\delta} f_\gamma f_\delta K_{\gamma\delta},$$

therefore

$$\frac{1}{|\,d\phi\,|^2} d(f\circ\phi)\,\mathrm{grad}\,\frac{|\,d(f\circ\phi)\,|^2}{|\,d\phi\,|^2}$$

$$= \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \partial_i \left(\frac{|\,d(f\circ\phi)\,|^2}{|\,d\phi\,|^2}\right)$$

$$= \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \partial_i \left(\sum_\beta f_\beta^2 K_\beta + \sum_{\delta\neq\gamma} f_\delta f_\gamma K_{\delta\gamma}\right)$$

$$= \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_\beta \partial_i\left(f_\beta^2\right) K_\beta + \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_\beta f_\beta^2 \partial_i\left(K_\beta\right)$$

$$+ \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_{\delta\neq\gamma} \partial_i\left(f_\delta f_\gamma\right) K_{\delta\gamma} + \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_{\delta\neq\gamma} f_\delta f_\gamma \partial_i\left(K_{\delta\gamma}\right)$$

$$= \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_{\beta,\delta} 2 f_\beta f_{\beta\delta}\phi_i^\delta K_\beta + \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_\beta f_\beta^2 \partial_i\left(K_\beta\right)$$

$$\frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_{\delta<\gamma} 2(f_\gamma f_{\mu\delta} + f_\delta f_{\mu\gamma})\phi_i^\mu K_{\delta\gamma} + \frac{1}{|\,d\phi\,|^2}\sum_{\alpha,i} f_\alpha \phi_i^\alpha \sum_{\delta\neq\gamma} f_\gamma f_\delta \partial_i\left(K_{\delta\gamma}\right)$$

$$= 2\sum_{\alpha,\beta,\delta} f_\alpha f_\beta f_{\beta\delta} K_{\alpha\delta} K_\beta + \frac{1}{|\,d\phi\,|^2}\sum_{\beta,\alpha,i} f_\alpha \phi_i^\alpha f_\beta^2 \partial_i\left(K_\beta\right)$$

$$2\sum_{\alpha,\mu,\delta<\gamma} f_\alpha(f_{\delta\mu} f_\gamma + f_\delta f_{\mu\gamma}) K_{\alpha\mu} K_{\delta\gamma} + \frac{1}{|\,d\phi\,|^2}\sum_{\beta,\alpha,i,\delta\neq\gamma} f_\alpha \phi_i^\alpha f_\delta f_\gamma \partial_i\left(K_{\delta\gamma}\right).$$

Using those identities in Equation (5) yields that

$$\tau_p(f\circ\phi) = |\,d\phi\,|^2 |\,d(f\circ\phi)\,|^{p-4}\left\{\left(\sum_\alpha f_\alpha^2 K_\alpha\right)\left(\sum_\alpha K_\alpha f_{\alpha\alpha} + \sum_{\alpha\neq\beta} K_{\alpha\beta} f_{\alpha\beta}\right) + \right. \tag{8}$$

$$\left(\frac{p-2}{2}\right)\left[2\sum_{\alpha,\beta,\delta} f_\alpha f_\beta f_{\beta\delta} K_{\alpha\delta} K_\beta + \frac{1}{|\,d\phi\,|^2}\sum_{\beta,\alpha,i} f_\alpha \phi_i^\alpha f_\beta^2 \partial_i\left(K_\beta\right) + \right.$$

$$\left.\left. 2\sum_{\alpha,\mu,\delta<\gamma} f_\alpha(f_{\delta\mu} f_\gamma + f_\delta f_{\mu\gamma}) K_{\alpha\mu} K_{\delta\gamma} + \frac{1}{|\,d\phi\,|^2}\sum_{\beta,\alpha,i,\delta\neq\gamma} f_\alpha \phi_i^\alpha f_\delta f_\gamma \partial_i\left(K_{\delta\gamma}\right)\right]\right\}.$$

Lemma 1.

$$\sum_{\alpha,i,\beta,\delta\neq\gamma} \tilde f_\alpha \phi_i^\alpha \left(\tilde f_\beta^2 \partial_i\left(K_\beta\right) + \tilde f_\delta \tilde f_\gamma \partial_i\left(K_{\delta\gamma}\right)\right) = 0, \quad \forall \tilde f \in C^1(N,\mathbb{R}).$$

Proof. By Proposition 2, we know we can find a local p-harmonic function f such that:

$$\frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}(q) = 0,$$
$$\frac{\partial f}{\partial y^\alpha}(q) = \tilde{f}_\alpha(q).$$

For this particular function, Equation (8) simplifies to

$$\left(\frac{p-2}{2}\right)\frac{1}{\mid d\phi\mid^2}\left(\sum_{\alpha,i,\beta,\delta\neq\gamma}\tilde{f}_\alpha\phi_i^\alpha\left(\tilde{f}_\beta^2\partial_i\left(K_\beta\right)+\tilde{f}_\delta\tilde{f}_\gamma\partial_i\left(K_{\delta\gamma}\right)\right)\right) = 0.$$

□

Lemma 2.

$$K_{\alpha\beta} = 0 \quad \forall\, \alpha \neq \beta.$$

Proof. As the choice of constants

$$C_\alpha = \delta_{\alpha\alpha_0}$$
$$C_{\alpha\beta} = \delta_{\alpha\alpha_0}^{\beta\beta_0} = \delta_{\alpha\alpha_0}\delta_{\beta\beta_0} \quad (\alpha_0 \neq \beta_0)$$

satisfies Condition (3) of Proposition 2, we can find a local p-harmonic function f in a neighbourhood of q with such first and second derivatives at the point q. For this function, Equation (8) reads

$$K_{\alpha_0}K_{\alpha_0\beta_0} + (p-2)\left(K_{\alpha_0\beta_0}K_{\alpha_0} + K_{\alpha_0\beta_0}K_{\alpha_0}\right) = 0. \tag{9}$$

Since $K_{\alpha_0} = 0$ implies $K_{\alpha_0\beta_0} = 0$, (9) yields

$$K_{\alpha_0\beta_0} = 0.$$

□

Lemma 3.

$$K_{\alpha_0} = K_{\beta_0} \quad \forall \alpha_0, \beta_0.$$

Proof. For any function f we can now write:

$$\tau(f\circ\phi) =\mid d\phi\mid^2\mid d(f\circ\phi)\mid^{p-4}\left\{\left(\sum_\alpha f_\alpha^2 K_\alpha\right)\left(\sum_\alpha K_\alpha f_{\alpha\alpha}\right) + (p-2)\left[\sum_{\alpha,\beta} f_\alpha f_\beta f_{\beta\alpha}K_\alpha K_\beta\right]\right\}.$$

Choose a local p-harmonic function f such that

$$f_{\alpha\beta}(q) = \delta_{\alpha\alpha_0}^{\beta\alpha_0} - \delta_{\alpha\beta_0}^{\beta\beta_0} \quad \alpha_0 \neq \beta_0$$
$$f_\alpha(q) = 1 \quad \forall\alpha$$

These coefficients do satisfy Condition (3), therefore

$$\begin{aligned}
\tau(f\circ\phi) =& \mid d\phi\mid^2 \mid d(f\circ\phi)\mid^{p-4}\left\{\left(\sum_\alpha K_\alpha\right)(K_{\alpha_0}-K_{\beta_0})+(p-2)\left(K_{\alpha_0}^2-K_{\beta_0}^2\right)\right\}\\
=& \ (K_{\alpha_0}-K_{\beta_0})\left(\sum_\alpha K_\alpha+(p-2)(K_{\alpha_0}+K_{\beta_0})\right)\\
=& \ 0.
\end{aligned}$$

Since $\sum_\alpha K_\alpha = 1$ and $(p-2)(K_{\alpha_0}+K_{\beta_0})\geq 0$, necessarily

$$K_{\alpha_0}=K_{\beta_0}.$$

□

We now complete the proof of Proposition 5:
Since $K_\beta = K$ (at a critical point $K=0$) and $K_{\alpha\beta}=0$ we have that:

$$\mid d\phi^\alpha\mid^2 = K \mid d\phi\mid^2,$$
$$< d\phi^\alpha, d\phi^\beta >= 0.$$

These two conditions can be written as one

$$\begin{aligned}
\lambda^2\delta^{\alpha\beta} &= \left\langle d\phi^\alpha, d\phi^\beta\right\rangle\\
&= \sum_{i,j=1}^m g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}
\end{aligned}$$

and this means precisely that ϕ is horizontally weakly conformal. □

Proposition 6. *Let $\phi:(M^m,g)\to(N^n,h)$ be a p-harmonic morphism then ϕ is a p-harmonic map.*

Proof. As functions are automatically horizontally weakly conformal and the following relation is valid for a horizontally weakly conformal map ϕ

$$\tau_p(f\circ\phi) = \frac{\mid df\mid^{p-2}}{n^{\frac{p-2}{2}}}df\left(\tau_p(\phi)\right)+\lambda^p\tau_p(f)$$

we see that a p-harmonic morphism mapping into a one-dimensional target manifold is a p-harmonic map.
We will now assume $n\geq 2$.
Fix $\alpha_0\in\{1,\ldots,n\}$. We will show the existence of a p-harmonic function f, in the neighbourhood of a given regular point $q\in M$, such that

$$\frac{\partial f}{\partial y^\alpha}(q)=\delta_{\alpha\alpha_0}.$$

The difficulty will be in finding the coefficients

$$C_{\alpha\beta}=\frac{\partial^2 f}{\partial y^\alpha\partial y^\beta}(q)$$

such that for this particular p-harmonic function, Equation (5) will read

$$\tau_p(f\circ\phi)=\theta^{p-2}\tau_p^{\alpha_0}(\phi)=0, \tag{10}$$

showing that ϕ is p-harmonic.
The coefficients $\{C_{\alpha\beta}\}$ will have to simultaneously satisfy Condition (3), which ensures the existence of f, and make the right-hand side part of Equation (5) vanish. First we recall the notations of the previous proposition:

$$\frac{|\, d\phi^\alpha\,|^2}{|\, d\phi\,|^2}=K_\alpha,$$
$$\frac{< d\phi^\alpha, d\phi^\beta >}{|\, d\phi\,|^2}=K_{\alpha\beta},$$

all the differentials being at the point q.
It is not difficult to see that if the $C_{\alpha\beta}$'s satisfy the following system (S) of two equations

(11)

$$\sum_\alpha C_{\alpha\alpha}+(p-2)C_{\alpha_0\alpha_0}=0$$

(12)

$$K_{\alpha_0}\left(\sum_\alpha K_\alpha C_{\alpha\alpha}+\sum_{\alpha\neq\beta}K_{\alpha\beta}C_{\alpha\beta}\right)+(p-2)\sum_{\mu\neq\alpha_0,\gamma}K_{\alpha_0\mu}C_{\mu\gamma}K_{\alpha_0\gamma}=H,$$

where $H=\frac{2-p}{|d\phi|^2}\sum_i\phi_i^{\alpha_0}\partial_i K_{\alpha_0}$, then they will define a local p-harmonic function f for which (10) stands.

- If there exists β_0 such that $K_{\alpha_0\beta_0}\neq 0$, then we can choose $C_{\alpha\alpha}=0,\forall\alpha$ and $C_{\alpha\beta}=0,\forall\alpha,\beta$ except for $C_{\alpha_0\beta_0}=\frac{H}{pK_{\alpha_0\beta_0}K_{\alpha_0}}$, as a solution to the system (S).
- If $K_{\alpha_0\beta}=0,\forall\beta\neq\alpha_0$ and there exists $\delta_0\neq\beta_0$ such that $K_{\delta_0\beta_0}\neq 0$, then the choice of $C_{\alpha\alpha}=0,\forall\alpha$ and $C_{\alpha\beta}=0,\forall\alpha,\beta$ except for $C_{\delta_0\beta_0}=\frac{H}{2K_{\delta_0\beta_0}K_{\alpha_0}}$, is a solution to the system (S).
- If $K_{\alpha\beta}=0,\forall\alpha\neq\beta$ but there exists β_0 such that $K_{\alpha_0}\neq K_{\beta_0}$, then we can take $C_{\alpha\alpha}=0,\forall\alpha$ except for $C_{\beta_0\beta_0}=\frac{H}{K_{\alpha_0}(K_{\beta_0}-K_{\alpha_0})}=(1-p)C_{\alpha_0\alpha_0}$ as solution to the system (S).
- Finally if $K_{\alpha\beta}=0,\forall\alpha\neq\beta$ and $K_\beta=K_{\alpha_0},\forall\beta$, then this means that the map ϕ is horizontally weakly conformal and, as remarked for the one-dimensional case, horizontally weakly conformal p-harmonic morphism are always p-harmonic maps.

$\square$

Combining the previous propositions we have

Theorem 2. *A map $\phi : (M^m, g) \to (N^n, h)$ is a p-harmonic morphism if and only if it is a horizontally weakly conformal p-harmonic map.*

Definition 7. A map $\phi : M \to N$ is called a *morphism of p-harmonic maps* if for any p-harmonic map $\psi : U \subseteq N \to P$, such that $\phi^{-1}(U) \neq \emptyset$,

$$\psi \circ \phi : \phi^{-1}(U) \subseteq M \to P$$

is a p-harmonic map.

Of course morphisms of p-harmonic maps are automatically morphisms of p-harmonic functions, i.e. p-harmonic morphisms.

Proposition 7. *A map $\phi : (M, g) \to (N, h)$ is a morphism of p-harmonic maps if and only if it is a horizontally weakly conformal p-harmonic map.*

Proof. The only thing to show is that a morphism of p-harmonic maps is a p-harmonic map. By definition of a morphism of p-harmonic maps, $\tau_p(\psi \circ \phi) = 0$ for any p-harmonic map ψ. However it is easy to see that the identity map of any Riemannian manifold is a p-harmonic map (since it is 2-harmonic and of constant norm) which proves that $\tau_p(\phi) = 0$. □

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