BIMINIMAL IMMERSIONS IN SPACE FORMS

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Abstract. We study biminimal immersions, that is immersions which are critical points of the bienergy for normal variations. We give a description of the Euler-Lagrange equation associated to biminimal immersions for: i) biminimal curves in a Riemannian manifold, with particular care to the case of curves in a space form ii) isometric immersions of codimension 1 in a Riemannian manifold, in particular for surfaces of a three dimensional space form. We describe two methods to construct biminimal surfaces in a space form using both Riemannian and horizontally homothetic submersions.

1. Introduction

A now classical result in Differential Geometry is that an isometric immersion is harmonic if and only if it is minimal [7]. This stems from the simple fact that the tension field is normal to the immage of the immersion and, therefore, characterises critical points of the energy functional, be it for variations normal to the image or all possible variations. Unfortunately this remarkable feature disappears when looking at the higher-degree problem of biharmonicity. Though constructed along the same lines as the energy functional, the Euler-Lagrange equations associated to the bienergy carry a tangential part, which goes a long way toward explaining the non-existence of (non-minimal) biharmonic immersed surfaces in $\mathbb{R}^3$ or $\mathbb{H}^3$ whether compact or not (see [5, 4] for details). The situation is hardly richer in $\mathbb{S}^3$, where $\mathbb{S}^2(\frac{1}{\sqrt{2}})$ is the sole possibilities [3].
To circumvent this obstacle, we propose to study isometric immersions which are critical points of the bienergy for normal variations. Note that this variational principle is close to the Willmore problem, the disparity being that we do not vary through isometric immersions.

Definition 1.1. A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is called *biharmonic* if it is a critical point, for all variations, of the bienergy functional:

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, v_g,$$

where $\tau(\phi) = \operatorname{trace} \nabla d\phi$ is the tension field.

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Definition 1.2. An immersion $\phi : (M^m, g) \to (N^n, h)$ $(m \leq n)$ between Riemannian manifolds, or its image, is called *biminimal* if it is a critical point of the bienergy functional E^2 for variations normal to the immage $\phi(M) \subset N$, that is:

$$\frac{dE_2(\phi_t)}{dt}\Big|_{t=0} = 0,$$

for any smooth variation of the map $\phi_t :]-\epsilon, +\epsilon[\times M \to N$, $\phi_0 = \phi$, such that $V = \frac{d\phi_t}{dt}|_{t=0}$ is normal to $\phi(M)$.

The Euler-Lagrange operator attached to biharmonicity, called the *bitension field* and computed by Jiang in [8], is:

$$\tau_2(\phi) = -\left(\Delta^{\phi}\tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi\right),$$

and vanishes if and only if the map ϕ is biharmonic. In the instance of an isometric immersion $\phi : M \to N$, requiring that the normal part of $\tau_2(\phi)$ is zero characterises biminimal maps, that is the mean curvature vector field $\mathbf{H}$ of ϕ satisfies:

$$[\Delta^{\phi}\mathbf{H} - \operatorname{trace} R^N(d\phi, \mathbf{H})d\phi]^{\perp} = 0. \tag{1}$$

While it is obvious that biharmonic immersions are biminimal, we will see in the following sections that the two notions are well distinct.

A generalisation of biminimal immersions are λ-*biminimal* immersions, they are defined as the critical points, with respect to normal variations of fixed energy, of the constrained bienergy functional

$$E_{2,\lambda}(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2\, v_g + \lambda \int_M |d\phi|^2\, v_g.$$

The Euler-Lagrange equation for λ-biminimal immersions is

$$[\tau_{2,\lambda}]^{\perp} = [\tau_2]^{\perp} - 2\lambda[\tau]^{\perp} = 0,$$

where $[,]^{\perp}$ denotes the normal component of $[,]$.

Notation. We shall place ourselves in the C^∞ category, i.e. manifolds, metrics, connections, maps will be assumed to be smooth. By (M^m, g) we shall mean a connected manifold, of dimension m, without boundary, endowed with a Riemannian metric g. We shall denote by ∇ the Levi-Civita connection on (M, g). For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. For the Laplacian we shall use $\Delta(f) = \operatorname{div}\operatorname{grad} f$ for functions $f \in C^\infty(M)$ and $\Delta^{\phi}W = -\operatorname{trace}\nabla\nabla^{\phi}W$ for sections along a map $\phi : M \to N$.

2. Biminimal curves

Our quest for examples of (non-minimal) biminimal submanifolds starts with curves.

Let $\gamma : I \subset \mathbb{R} \to (M^m, g)$ be a curve parametrised by arc-length in a Riemannian manifold (M^m, g), that is γ is an isometric immersion. Before computing the bitension field of γ we recall the definition of the Frenet frame.

Definition 2.1 (See, for example, [9]). The Frenet frame $\{B_i\}_{i=1,\dots,m}$ associated to a curve $\gamma : I \subset \mathbb{R} \to (M^m, g)$ is the orthonormalisation of the m-uple $\{\nabla^{(k)}_{\frac{\partial}{\partial t}} d\gamma(\frac{\partial}{\partial t})\}_{k=0,\dots,m-1}$, described by:

$$
\begin{aligned}
B_1 &= d\gamma(\frac{\partial}{\partial t}),\\
\nabla^{\gamma}_{\frac{\partial}{\partial t}} B_1 &= k_1 B_2,\\
\nabla^{\gamma}_{\frac{\partial}{\partial t}} B_i &= -k_{i-1} B_{i-1} + k_i B_{i+1}, \quad \forall i = 2, \dots, m-1,\\
\nabla^{\gamma}_{\frac{\partial}{\partial t}} B_m &= -k_{m-1} B_{m-1},
\end{aligned}
$$

where the functions $\{k_1 = k > 0, k_2 = \tau, k_3, \dots, k_{m-1}\}$ are called the curvatures of γ. Note that $B_1 = T = \gamma'$ is the unit tangent vector field to the curve.
In the instance of a curve γ on a surface ($m = 2$), the Frenet frame reduces to the couple $\{T, N\}$, T being the unit tangent vector field along γ and N a normal vector field along γ such that $\{T, N\}$ is a positive bases, while $k_1 = k$ is the signed curvature of γ.

We are now ready to characterise biminimal curves in a Riemannian manifold.

Proposition 2.1. *Let $\gamma : I \subset \mathbb{R} \to (M^m, g)$ $(m \geq 2)$ be an isometric curve from an open interval of $\mathbb{R}$ into a Riemannian manifold (M, g). Then γ is biminimal if and only if:*

$$
(2) \qquad \begin{cases}
k_1'' - k_1^3 - k_1 k_2^2 + k_1 g(R(B_1, B_2)B_1, B_2) = 0,\\
(k_1^2 k_2)' + k_1^2 g(R(B_1, B_2)B_1, B_3) = 0,\\
k_1 k_3 + k_1 g(R(B_1, B_2)B_1, B_4) = 0,\\
k_1 g(R(B_1, B_2)B_1, B_j) = 0, \quad j = 5, \dots, m,
\end{cases}
$$

where R is the curvature tensor of (M, g) and $\{B_i\}_{i=1,\dots,m}$ the Frenet frame of γ.

Proof. With respect to its Frenet frame, the tension field of γ is:

$$
\tau(\gamma) = \operatorname{trace} \nabla d\gamma = \nabla^{\gamma}_{\frac{\partial}{\partial t}}(d\gamma(\frac{\partial}{\partial t})) - d\gamma(\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}) = \nabla^{\gamma}_{\frac{\partial}{\partial t}} B_1 = k_1 B_2
$$

and its bitension field:

$$
\begin{aligned}
\tau_2(\gamma) =& -\nabla^{\gamma}_{\frac{\partial}{\partial t}} \nabla^{\gamma}_{\frac{\partial}{\partial t}}(\tau(\gamma)) + \nabla^{\gamma}_{\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}}(\tau(\gamma)) - R(d\gamma(\frac{\partial}{\partial t}), \tau(\gamma)) d\gamma(\frac{\partial}{\partial t})\\
=& -\nabla^{\gamma}_{\frac{\partial}{\partial t}} \nabla^{\gamma}_{\frac{\partial}{\partial t}}(k_1 B_2) - R(B_1, k_1 B_2) B_1\\
=& -\nabla^{\gamma}_{\frac{\partial}{\partial t}}(k_1' B_2 - k_1^2 B_1 + k_1 k_2 B_3) - k_1 R(B_1, B_2) B_1\\
=& -(k_1'' - k_1^3 - k_1 k_2^2) B_2 + 3 k_1 k_1' B_1 - (k_1' k_2 + (k_1 k_2)') B_3\\
& - k_1 k_3 B_4 - k_1 R(B_1, B_2) B_1.
\end{aligned}
$$

The vanishing of the normal components yields System (2). □

Remark 2.1. Asking γ to be biharmonic requires the supplementary condition $[\tau_2(\gamma)]^{B_1} = 0$, equivalent to $k_1 k_1' = 0$, that is, either k_1 is constant or γ is a geodesic ($k_1 = 0$).

If the target manifold is surface or a three dimensional Riemannian manifold with constant sectional curvature Equations (2) are more manageable as shown in the following.

Corollary 2.1.

i) *An isometric curve γ on a surface of Gaussian curvature G is biminimal if and only if its signed curvature k satisfies the ordinary differential equation:*

$$k'' - k^3 + kG = 0. \tag{3}$$

ii) *An isometric curve γ on a Riemannian three-manifold of constant sectional curvature c is biminimal if and only if its curvature k and torsion τ fulfill the system:*

$$\begin{cases} k'' - k^3 - kt^2 + kc = 0 \\ k^2 t = constant \end{cases} \tag{4}$$

Proof. For i). The two-dimensional Frenet frame of γ consists only of T and N, and the curve is biminimal if and only if :

$$k'' + k^3 + kg(R(T,N)T,N) = 0,$$

but since $g(R(T,N)T,N) = G$, we obtain (3).
In dimension three, the Frenet frame of γ is $\{T, N = B_2, B = B_3\}$, and the conditions of Proposition 2.1 become:

$$\begin{cases} k'' - k^3 - kt^2 + kg(R(T,N)T,N) = 0, \\ (k^2t)' + k^2 g(R(T,N)T,B) = 0. \end{cases}$$

The constant sectional curvature of the target means that $g(R(T,N)T,N) = c$ and $g(R(T,N)T,B) = 0$. $\square$

From Corollary 2.1 if γ is an isometric curve in a Riemannian manifold M^n of constant sectional curvature c and dimension $n = 2, 3$, then the curvature of γ (the signed curvature when $n = 2$), satisfies the equation

$$k'' - k^3 - \frac{\alpha^2}{k^3} + kc = 0, \tag{5}$$

where $\alpha = k^2\tau$.
Multiplying Equation (5) by $2k'$ and integrating, we obtain:

$$(k')^2 - k^4/2 + \frac{\alpha^2}{k^2} + ck^2 = A, \qquad A \in \mathbb{R},$$

and setting $u = k^2$ yields:

$$(u')^2 - 2u^3 + 4\alpha^2 + 4cu^2 = 4Au.$$

Since this equation is of the form $(u')^2 = P(u)$, P being a polynomial of degree three, it can be solved by standard techniques in terms of elliptic functions (see, for example

[6]). In a forthcoming paper we shall give an accurate description of the solutions of Equation (5). Here we just point out that if M is the flat $\mathbb{R}^2$, then Equation (5) reduces to

$$k'' - k^3 = 0,$$

of which one solution can be expressed in terms of elementary functions, that is $k(s) = \sqrt{2}/s$, where s is the arc-length. Now using the standard formula to integrate the curve of known signed curvature, we find that, up to isometry of $\mathbb{R}^2$, this biminimal curve is given by

$$\begin{aligned}\gamma(s) = \quad & s/3\{\cos(\sqrt{2}\log(s)) + \sqrt{2}\sin(\sqrt{2}\log(s)), \\ & -\sqrt{2}\cos(\sqrt{2}\log(s)) + \sin(\sqrt{2}\log(s))\}.\end{aligned}$$

This is the standard parametrisation by arc-length of the logarithmic spiral plotted in Figure 1.

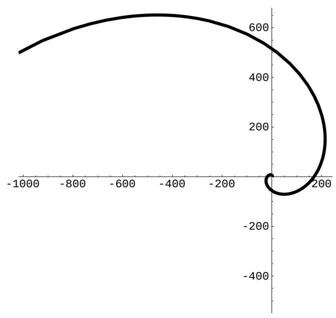

FIGURE 1. Biminimal curve in $\mathbb{R}^2$ of logarithmic type.

2.1. **Biminimal curves via conformal changes of the metric.** Any representative in the conformal class $[g]$ of a Riemannian metric g on a manifold M, can be expressed as $\overline{g} = e^{2f}g$, $f \in C^\infty(M)$, and their Levi-Civita connections are related, for $X, Y \in C(TM)$, by (cf. [2]):

$$\overline{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X,Y)\,\mathrm{grad}\, f. \tag{6}$$

Observe that a geodesic curve γ on (M, g) will not remain a geodesic after a conformal change of the metric, unless the conformal factor is constant since:

$$\overline{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + 2\dot\gamma(f)\dot\gamma - |\dot\gamma|^2\,\mathrm{grad}\, f.$$

The following theorem gives a tool to construct examples of biminimal curves.

Theorem 2.1. *Let (M^m, g) be a Riemannian manifold. Fix a point $p \in M^m$ and choose a function f depending only on the geodesic distance from p. Any geodesic on (M, g) going through p is biminimal on $(M^m, \overline{g} = e^{2f}g)$.*

Proof. Let γ be a geodesic of (M^m, g) and let $\{B_i\}_{i=1,\dots,m}$ be the Frenet frame associated to the curve γ (cf. Definition 2.1). Since the function f depends only on the geodesic distance from p, i.e. the B_1-direction, $B_i f = 0$, $\forall i = 2, \dots, m$. Since γ is a geodesic curve on (M, g), then:

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} B_1 = 0,$$

and the tension field of γ with respect to the metric $\overline{g} = e^{2f} g$ is:

$$\overline{\tau}(\phi) = \overline{\nabla}^{\gamma}_{\frac{\partial}{\partial t}} d\gamma(\frac{\partial}{\partial t}) = \overline{\nabla}^{\gamma}_{\frac{\partial}{\partial t}} B_1 = \nabla^{\gamma}_{\frac{\partial}{\partial t}} B_1 + 2B_1(f)B_1 - \operatorname{grad} f,$$

where in the last equality we have used (6) linking the two Levi-Civita connections. Besides:

$$\operatorname{grad} f = B_1(f)B_1 + \sum_{i=2}^{m} B_i(f)B_i = B_1(f)B_1,$$

thus $\overline{\tau}(\phi) = B_1(f)B_1$.
Still with respect to $\overline{g}$, the bitension field of γ is:

$$\begin{aligned}
\overline{\tau}^2(\phi) &= -\Delta^{\gamma}\overline{\tau}(\phi) - \operatorname{trace} \overline{R}(d\gamma, \overline{\tau}(\phi))d\gamma \\
&= \overline{\nabla}^{\gamma}_{\frac{\partial}{\partial t}} \overline{\nabla}^{\gamma}_{\frac{\partial}{\partial t}} (B_1(f)B_1) - \overline{\nabla}^{\gamma}_{\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}} (B_1(f)B_1) - \overline{R}(B_1, B_1(f)B_1)B_1 \\
&= \overline{\nabla}^{\gamma}_{\frac{\partial}{\partial t}} \overline{\nabla}^{\gamma}_{\frac{\partial}{\partial t}} (B_1(f)B_1) \\
&= \overline{\nabla}^{\gamma}_{\frac{\partial}{\partial t}} (B_1B_1(f)B_1 + B_1(f)^2 B_1) \\
&= (B_1B_1B_1(f))B_1 + B_1B_1(f)B_1(f)B_1 + 2B_1B_1(f)B_1(f)B_1 + (B_1(f))^3 B_1 \\
&= [B_1B_1B_1(f) + 3B_1B_1(f)B_1(f) + (B_1(f))^3]B_1.
\end{aligned}$$

So $\overline{\tau}^2(\phi)$ has no normal component and γ is biminimal on $(M^m, \overline{g} = e^{2f}g)$. □

Corollary 2.2. *Let r be the geodesic distance from a point $p \in (M, g)$, and $f(r) = \ln(ar^2 + br + c)$, $a, b, c \in \mathbb{R}$. Then a geodesic on (M, g) through p becomes a biharmonic map on $(M, \overline{g} = e^{2f}g)$.*

Proof. From the proof of Theorem 2.1 a geodesic on (M, g) through p becomes a biharmonic map on $(M, \overline{g})$ if f is a solution of the ordinary differential equation

$$f'''(r) + 3f''(r)f'(r) + f'(r)^3 = 0.$$

To solve this equation put $y = f'$ to obtain

$$y'' + 3y'y + y^3 = 0. \tag{7}$$

Then, using the transformation $y = x'/x$ Equation (7) becomes $x'''/x = 0$ which has the solution $x(r) = \bar{a}r^2 + \bar{b}r + \bar{c}$, $\bar{a}, \bar{b}, \bar{c} \in \mathbb{R}$. Finally, from $f(r) = \ln(d\, x(r))$, $d \in \mathbb{R}$, we find the desired f. □

As an example, one can take $(M,g) = (\mathbb{R}^2, g = dx^2 + dy^2)$, and $f(r) = \ln(r^2+1)$, where $r = \sqrt{x^2+y^2}$ is the distance from the origin. Thus any straight line on the flat $\mathbb{R}^2$ turns to a biharmonic curve on $(\mathbb{R}^2, \bar{g} = (r^2+1)^2(dx^2+dy^2))$ which is the metric, in local isothermal coordinates, of the Enneper minimal surface. In Figure 2 there is a plot of Enneper surface in polar coordinates, so that the radial curves on the picture are biharmonic.

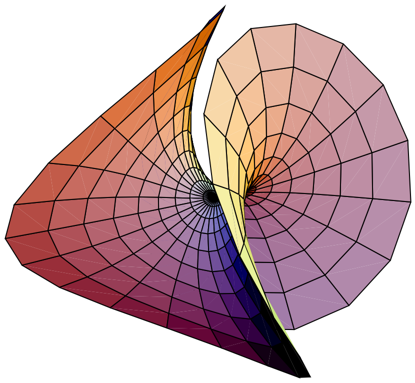

FIGURE 2. The radial curves from the origin of the Enneper surface are biharmonic.

Remark 2.2. One should notice that though it is possible to take closed curves in Theorem 2.1, they then fail to be isometric immersions

3. Codimension-one biminimal submanifolds

Let $\phi : M^n \to N^{n+1}$ be an isometric immersion of codimension-one. We denote by B the second fundamental form of ϕ, by $\mathbf{N}$ a unit normal vector field to $\phi(M) \subset N$ and by $\mathbf{H} = H\mathbf{N}$ the mean curvature vector field of ϕ (H the mean curvature function).
Then we have

Proposition 3.1. *Let $\phi : M^n \to N^{n+1}$ be an isometric immersion of codimension-one and $\mathbf{H} = H\mathbf{N}$ its mean curvature vector. Then ϕ is biminimal if and only if:*

$$\Delta H = (|B|^2 - \operatorname{Ricci}(\mathbf{N}))H. \tag{8}$$

Proof. In a local orthonormal frame $\{e_i\}_{i=1,\dots,n}$ on M, the tension field of ϕ is $\tau(\phi)=nH\mathbf{N}$ and its bitension field is:

$$\begin{aligned}
-\tau_2(\phi)=&-\sum_{i=1}^{n}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}(nH\mathbf{N})+\nabla^{\phi}_{\nabla_{e_i}e_i}(nH\mathbf{N})-\\
&-\sum_{i=1}^{n}R^{N^{n+1}}(d\phi(e_i),nH\mathbf{N})d\phi(e_i)\\
=&n\sum_{i=1}^{n}\Big[-\nabla^{\phi}_{e_i}\left(e_i(H)\mathbf{N}+H\nabla^{\phi}_{e_i}\mathbf{N}\right)+(\nabla_{e_i}e_i)(H)\mathbf{N}+H\nabla^{\phi}_{\nabla_{e_i}e_i}\mathbf{N}-\\
&-HR^{N^{n+1}}(d\phi(e_i),\mathbf{N})d\phi(e_i)\Big]\\
=&n\sum_{i=1}^{n}\Big[-e_ie_i(H)\mathbf{N}-2e_i(H)\nabla^{\phi}_{e_i}\mathbf{N}-H\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\mathbf{N}+(\nabla_{e_i}e_i)(H)\mathbf{N}+\\
&+H\nabla^{\phi}_{\nabla_{e_i}e_i}\mathbf{N}\Big]-nH\sum_{i=1}^{n}R^{N^{n+1}}(d\phi(e_i),\mathbf{N})d\phi(e_i)\\
=&n(\Delta H)\mathbf{N}-2n\sum_{i=1}^{n}e_i(H)\nabla^{\phi}_{e_i}\mathbf{N}+nH\Delta^{\phi}\mathbf{N}-\\
&-nH\sum_{i=1}^{n}R^{N^{n+1}}(d\phi(e_i),\mathbf{N})d\phi(e_i)
\end{aligned}$$

But

i) $<\nabla^{\phi}_{e_i}\mathbf{N},\mathbf{N}>=\frac{1}{2}e_i<\mathbf{N},\mathbf{N}>=0$;
ii) $<\sum_{i=1}^{n}R^{N^{n+1}}(d\phi(e_i),\mathbf{N})d\phi(e_i),\mathbf{N}>=\mathrm{Ricci}(\mathbf{N})$.

For $<\Delta^{\phi}\mathbf{N},\mathbf{N}>$, first we have that

$$<\Delta^{\phi}\mathbf{N},\mathbf{N}>=\sum_{i=1}^{n}<-\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\mathbf{N}+\nabla^{\phi}_{\nabla_{e_i}e_i}\mathbf{N},\mathbf{N}>=\sum_{i=1}^{n}<\nabla^{\phi}_{e_i}\mathbf{N},\nabla^{\phi}_{e_i}\mathbf{N}>.$$

Then if B is the second fundamental form of ϕ, which, in an orthonormal frame $\{e_1,\dots,e_n,\mathbf{N}\}$, is defined by:

$$B=(<\nabla_{e_i}e_j,\mathbf{N}>)_{i,j=1,\dots,n}=-(<\nabla_{e_i}\mathbf{N},e_j>)_{i,j=1,\dots,n}$$

we have

$$|\nabla^{\phi}_{e_i}\mathbf{N}|^2=\left\langle\nabla^{\phi}_{e_i}\mathbf{N},\nabla^{\phi}_{e_i}\mathbf{N}\right\rangle=\sum_{j=1}^{n}\left\langle\nabla^{\phi}_{e_i}\mathbf{N},e_j\right\rangle^2\quad(\forall i=1,\dots,n),$$

which implies that

$$\sum_{i=1}^{n}<\nabla^{\phi}_{e_i}\mathbf{N},\nabla^{\phi}_{e_i}\mathbf{N}>=|B|^2.$$

In conclusion:

$$-\langle\tau_2(\phi),\mathbf{N}\rangle=n\left(-\Delta H+H|B|^2-H\,\mathrm{Ricci}(\mathbf{N})\right).$$

□

Corollary 3.1. *An isometric immersion $\phi : M^n \to N^{n+1}(c)$ into a space form of constant curvature c is biminimal if and only if:*

$$\Delta H - H(n^2H^2 - s + n(n-2)c) = 0,$$

where H is the mean curvature and s the scalar curvature of M^n. Moreover, an isometric immersion $\phi : M^2 \to N^3(c)$ from a surface to a three dimensional space form is λ-biminimal if and only if:

$$\Delta H - 2H(2H^2 - G + \lambda) = 0, \tag{9}$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame of M^n corresponding to the principal curvatures $\{k_1, \dots, k_n\}$ and B its second fundamental form, then:

$$\begin{aligned}
|B|^2 = k_1^2 + \cdots + k_n^2 &= n^2H^2 - 2\sum_{\substack{i,j=1\\ i<j}}^{n} k_ik_j \\
&= n^2H^2 - 2\sum_{\substack{i,j=1\\ i<j}}^{n} (K(e_i,e_j) - c) = n^2H^2 - \sum_{i,j=1}^{n} K(e_i,e_j) + n(n-1)c \\
&= n^2H^2 - s + n(n-1)c,
\end{aligned}$$

where $K(e_i, e_j)$ is the sectional curvature of the plane spanned by e_i and e_j, and $s = \sum_{i,j=1}^{n} K(e_i, e_j)$ is the scalar curvature of M^n. Since $\mathrm{Ricci}(\mathbf{N}) = nc$, the map ϕ is biminimal if and only if:

$$\Delta H = (n^2H^2 - s + n(n-2)c)H.$$

□

Remark 3.1. Condition (9) is very similar to the equation of the Willmore problem ($\Delta H + 2H(H^2 - K) = 0$) but the minus sign in (9) rules out the existence of compact solutions when $c \leq 0$.

We shall now describe some constructions to produce examples of biminimal immersions. For it, let recall that a submersion $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is a *horizontally homothetic submersion* if there exists a function $\lambda : M \to \mathbb{R}$, called the *dilation function*, such that:

- at each point $p \in M$ the differential $d\phi_p : H_p \to T_{\phi(p)}N$ is a conformal map with factor $\lambda(p)$, i.e. $\lambda^2(p)g(X,Y)(p) = h(d\phi_p(X), d\phi_p(Y))(\phi(p))$ for all $X, Y \in H_p = Ker_p(d\phi)^{\perp}$;
- $X(\lambda^2) = 0$ for all horizontal vector fields, X being horizontal if $X_p \in H_p$ for all $p \in M$.

Lemma 3.1. *Let $\phi : (M^n, g) \to (N^2, h)$ be a horizontally homothetic submersion with dilation λ and minimal fibres and let $\gamma : I \subset \mathbb{R} \to N^2$ be a curve parametrised by arc-length of signed curvature k_γ. Then the codimension-one submanifold $S = \phi^{-1}(\gamma(I)) \subset M$ has mean curvature $H_S = \lambda k_\gamma / n$.*

Proof. Let $\{T, N\}$ be the Frenet frame of γ, i.e. $\nabla^\gamma_{\frac{\partial}{\partial t}} T = k_\gamma N$. Choose a local orthogonal frame $\{e_1, e_2, e_3, \dots, e_n\}$ on M^n such that $d\phi(e_1) = T \circ \phi$, $d\phi(e_2) = N \circ \phi$ and $d\phi(e_i) = 0$, for $i = 3, \dots, n$. Since ϕ is a horizontally homothetic submersion we have that $|e_1|^2 = |e_2|^2 = 1/\lambda^2$ and we can choose $\{e_3, \dots, e_n\}$ of unit length. The restriction on $S = \phi^{-1}(\gamma(I)) \subset M$ of the vector fields e_1 and $\{e_3, \dots, e_n\}$ give a local frame of tangent vector fields to the submanifold $S = \phi^{-1}(\gamma(I))$, while the restriction of λe_2 gives a unit normal vector field on S. Therefore the mean curvature of S is:

$$H_S = \frac{1}{n}\lambda^3 \langle \nabla_{e_1} e_1, e_2 \rangle + \frac{1}{n}\lambda \sum_{i=3}^{n} \langle \nabla_{e_i} e_i, e_2 \rangle = \frac{1}{n}\lambda^3 \langle \nabla_{e_1} e_1, e_2 \rangle + \frac{n-2}{n}\lambda(H_{\text{fibre}}),$$

where H_{fibre} is the mean curvature of the fibres. Since the fibres are minimal ($H_{\text{fibre}} = 0$) we get

$$H_S = \frac{1}{n}\lambda^3 < \nabla_{e_1} e_1, e_2 > .$$

Moreover:

$$\begin{aligned}
\lambda^2 < \nabla_{e_1} e_1, e_2 > &=< d\phi(\nabla_{e_1} e_1), d\phi(e_2) > \\
&=< d\phi(\nabla_{e_1} e_1), N \circ \phi > \\
&=< \nabla^\phi_{e_1} d\phi(e_1), N \circ \phi > \\
&=< \nabla^\phi_{e_1} (T \circ \phi), N \circ \phi >,
\end{aligned}$$

since $(\nabla d\phi)(e_1, e_1) = 0$ for a horizontally homothetic submersion (cf. [1]).
Finally,

$$\nabla^\phi_{e_1}(T \circ \phi) = (\nabla_{d\phi(e_1)} T) \circ \phi = (\nabla_T T) \circ \phi,$$

and $nH_S = \lambda k_\gamma$. □

Theorem 3.1. *Let $\phi : M^3(c) \to (N^2, h)$ be a horizontally homothetic submersion with dilation λ, minimal fibres and integrable horizontal distribution from a space form of constant sectional curvature c to a surface. Let $\gamma : I \subset \mathbb{R} \to N^2$ be curve parametrised by arc-length such that the surface $S = \phi^{-1}(\gamma) \subset M^3$ has constant Gaussian curvature c. Then $S = \phi^{-1}(\gamma) \subset M^3$ is a c-biminimal surface if and only if γ is a biminimal curve.*

Proof. Let $\{T, N\}$ be the Frenet frame of γ, i.e. $\nabla_T T = k_\gamma N$. Let $\tilde{\gamma} : I \to M^3$ be a horizontal lift of γ, so that $\tilde{\gamma}'$ is horizontal and $d\phi(\tilde{\gamma}') = T \circ \phi$. Let $\psi(t, s) = \eta_s(\tilde{\gamma}(t))$ be a local parametrisation of the surface $S = \phi^{-1}(\gamma) \subset M^3$, where, for a fixed $t_0 \in I$, $\eta_s(\tilde{\gamma}(t_0))$

is a parametrisation by arc-length of the fibre of ϕ through $\tilde{\gamma}(t_0)$. Then ψ induces on the surface S the metric

$$g_S = \frac{1}{\lambda^2}dt^2 + ds^2,$$

where λ is the dilation of ϕ which, restricted to the surface S, depends only on s. The Laplacian on S is then given by

$$\Delta = \lambda^2\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2} - \operatorname{grad}(\log(\lambda)\frac{\partial}{\partial s}, \tag{10}$$

whilst the Gaussian curvature of S reduces to

$$G_S = \frac{\Delta\lambda}{\lambda} - (\operatorname{grad}(\log(\lambda))^2. \tag{11}$$

Now, assuming that S has constant Gaussian curvature $G_S = c$, from Equation(9), S is c-biminimal in $M^3(c)$ if and only if

$$\Delta H - 2H(2H^2 - c + c) = \Delta H - 4H^3 = 0.$$

From Lemma 3.1, $2H = \lambda k_\gamma$, thus

$$\begin{aligned} 2(\Delta H - 4H^3) =& \Delta k_\gamma - k_\gamma^3 \\ =& \lambda^3\left[k_\gamma'' - k_\gamma^3 + \frac{k_\gamma}{\lambda^2}\left(G_S + (\operatorname{grad}(\log(\lambda))^2\right)\right] \\ =& \lambda^3\left[k_\gamma'' - k_\gamma^3 + \frac{k_\gamma}{\lambda^2}\left(c + (\operatorname{grad}(\log(\lambda))^2\right)\right] \end{aligned} \tag{12}$$

Finally, from the generalised O'Neil formula relating the sectional curvatures of the domain and of the target manifolds for a given horizontally homothetic submersion with integrable horizontal distribution (see, for example [1, Corollary 11.2.3]) we get

$$\frac{1}{\lambda^2}\left(c + (\operatorname{grad}(\log(\lambda))^2\right) = G_N,$$

which, together with (12), gives

$$2(\Delta H - 4H^3) = \lambda^3(k_\gamma'' - k_\gamma^3 + k_\gamma G_N).$$

Then the theorem follows from Corollary 2.1. □

If the horizontal space is not integrable, we can reformulate Theorem 3.1, for Riemannian submersions.

Theorem 3.2. *Let $\phi : M^3(c) \to N^2(\bar{c})$ be a Riemannian submersion with minimal fibres from a space form of constant sectional curvature c to a surface of constant Gaussian curvature $\bar{c}$. Let $\gamma : I \subset \mathbb{R} \to N^2$ be a curve parametrised by arc-length. Then $S = \phi^{-1}(\gamma) \subset M^3$ is a biminimal surface if and only if γ is a $\bar{c}$-biminimal curve.*

Proof. First from (10) and (11), since $\lambda = 1$, we have

$$G_S = 0$$
$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}.$$

Thus, taking into account Lemma 3.1, S is biminimal if and only if

$$\Delta(2H) - (2H)^3 = k_\gamma'' - k_\gamma^3 = 0.$$

From Corollary 2.1 the latter equation is clearly the $\bar{c}$-biminimal equation for the curve $\gamma : I \to N^2(\bar{c})$.

□

3.1. **Examples of biminimal surfaces in $\mathbb{R}^3$.** We apply Theorem 3.1 to construct examples of biminimal surfaces in $\mathbb{R}^3$ with the flat metric.

1. First we consider the orthogonal projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, given by $\pi(x, y, z) = (x, y)$. The projection π is clearly a Riemannian submersion with minimal fibres (vertical straight lines of $\mathbb{R}^3$) and integrable horizontal distribution. Thus from Theorem 3.1 a vertical cylinders with generatrix a biminimal curve of $\mathbb{R}^2$ is a biminimal surfaces. For example one can consider the cylinder on the logarithmic spiral.
2. The space $\mathbb{R}^3 \setminus \{0\}$ can be described as the warped product $\mathbb{R}^3 \setminus \{0\} = \mathbb{R}^+ \times_{t^2} \mathbb{S}^2$ with the warped metric $g = dt^2 + t^2 d\theta^2$, $d\theta^2$ being the canonical metric on $\mathbb{S}^2$. Then the projection to the second factor $\pi_2 : \mathbb{R}^+ \times_{t^2} \mathbb{S}^2 \to \mathbb{S}^2$ is a horizontally homothetic submersion with dilation $1/t$, integrable horizontal distribution and minimal fibres. Geometrically π_2 is the radial projection $p \mapsto p/|p|$, $p \in \mathbb{R}^3 \setminus \{0\}$. Applying again Theorem 3.1, we see that the cone on a biminimal curve on $\mathbb{S}^2$ is a biminimal surface of $\mathbb{R}^3$. For example if we take the parallel on $\mathbb{S}^2$ of latitude $\pi/4$, which is a biharmonic curve and thus biminimal, we get the standard cone of revolution in $\mathbb{R}^3$.
3. The following example does not seem to enter the picture of Theorem 3.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a space curve with curvature k equal to its torsion τ and let $\{T, N, B\}$ be its Frenet frame. It is easy to see that the envelope S of γ, parametrised by: $X(u, s) = \alpha(s) + u(B + T)$, has mean curvature $H = k$. Thus S is biminimal if and only if:

$$\Delta H - 4H^3 = k'' - 4k^3 = 0. \tag{13}$$

Geometrically the curve γ is a curve with constant slope, i.e. there exists a constant vector $u \in \mathbb{R}^3$ such that $< T, u >=$ constant. Then γ can be described as an helix of the cylinder on a plane curve β (the orthogonal projection of γ to a plane orthogonal to u) whose geodesic curvature is a solution of (13). For example we can take β to be the logarithmic spiral of natural equation $k_\beta = 1/(\sqrt{2}s)$.

3.2. **Examples of (-1)-biminimal surfaces in $\mathbb{H}^3$.**

1. Let $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \ : \ z > 0\}$ be the half space model for the hyperbolic space endowed with the metric of constant sectional curvature 1 given by $g = (dx^2 + dy^2 + dz^2)/z^2$. Then the projection to the plane at infinity defines a horizontally homothetic submersion $\pi : \mathbb{H}^3 \to \mathbb{R}^2$ with dilation $\lambda = z$, integrable horizontal distribution and minimal fibres (vertical lines in $\mathbb{H}^3$). Then, from Theorem 3.1, a vertical cylinder with generatrix a biminimal curve of $\mathbb{R}^2$ is a (-1)-biminimal surface in the hyperbolic space. For example the cylinder on the logarithmic spiral is biminimal in $\mathbb{R}^3$ whilst it is (-1)-biminimal in $\mathbb{H}^3$.
2. Let $\pi : \mathbb{H}^3 \to \mathbb{H}^2$ defined by $\pi(x, y, z) = (x, 0, \sqrt{y^2 + z^2})$. The fibres of π over $(x, 0, r)$ is the semicircle with centre $(x, 0, r)$, radius r, and parallel to the coordinate yz-plane. Thus the map π has minimal fibres. Geometrically this map is a projection along the geodesics of $\mathbb{H}^3$ which are orthogonal to $\mathbb{H}^2$. This is again a horizontally homothetic submersion with dilation, along the fibres, $\lambda(s) = 1/\cosh(s)$, s being the arc-length parameter of the fibre. An easy computation shows that for any curve γ parametrised by arc-length in $\mathbb{H}^2$ the surface $S = \pi^{-1}(\gamma)$ is of constant Gaussian curvature -1. Then, applying Theorem 3.1, for any biminimal curve of $\mathbb{H}^2$ $S = \pi^{-1}(\gamma)$ is a (-1)-biminimal surface in the hyperbolic space.

3.3. **Examples of λ-biminimal surfaces in $\mathbb{S}^3$.**

1. Let $p, q \in \mathbb{S}^3$ be two antipodal points. Then the space $\mathbb{S}^3 \setminus \{p, q\}$ can be described as the warped product $\mathbb{S}^3 \setminus \{p, q\} = (0, \pi) \times_{\sin^2(t)} \mathbb{S}^2$ with the warped metric $g = dt^2 + \sin^2(t) d\theta^2$, $d\theta^2$ being the canonical metric on $\mathbb{S}^2$. Then the projection to the second factor $\pi_2 : \mathbb{R}^+ \times_{t^2} \mathbb{S}^2 \to \mathbb{S}^2$ is a horizontally homothetic submersion with dilation $1/\sin(t)$, integrable horizontal distribution and minimal fibres. Geometrically π_2 is the projection along the longitudes onto the equatorial sphere. Theorem 3.1 gives a correspondence between biminimal curves on $\mathbb{S}^2$ and 1-biminimal surfaces of $\mathbb{S}^3$ given by $S = \pi_2^{-1}(\gamma)$.
2. This is the only example for which we use Theorem 3.2. Let $H : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$ be the Hopf map defined by $H(z, w) = (2z\bar{w}, |z|^2 - |w|^2)$, where we have identified $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \ : \ |z|^2 + |w|^2 = 1\}$ and $\mathbb{S}^2(1/2) = \{(z, t) \in \mathbb{C} \times \mathbb{R} \ : \ |z|^2 + t^2 = 1/4\}$. The Hopf map is a Riemannian submersion with minimal fibres (great circles in $\mathbb{S}^3$). Moreover, it is a well known fact that the Hopf cylinder $H^{-1}(\gamma)$, γ being a curve in $\mathbb{S}^2(1/2)$ has zero Gaussian curvature. Thus, from Theorem 3.2, we see that a Hopf cylinder is a biminimal surface of $\mathbb{S}^3$ if and only if the curve γ is a 4-biminimal curve of $\mathbb{S}^2(1/2)$. This link between biminimal Hopf cylinders and biminimal curves on $\mathbb{S}^2$ is similar to the link between Willmore Hopf cylinders and elastic curves on $\mathbb{S}^2$ proved by U. Pinkall in [11].

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