On singular perturbations for differential inclusions on the infinite interval

F. Watbled*

Abstract. We consider a differential inclusion subject to a singular perturbation, i.e. part of the derivatives are multiplied by a small parameter $\varepsilon > 0$. We show that under some stability and structural assumptions, every solution of the singularly perturbed inclusion comes close to a solution of the degenerate inclusion (obtained for $\varepsilon = 0$) when ε tends to 0. The goal of the present paper is to provide a new result of Tikhonov type on the time interval $[0, +\infty[$.

Key words . Singular perturbation, differential inclusions, asymptotic stability, Liapunov functions, Tikhonov's theorem.
AMS Subject Classifications. 34A60, 34E15

Introduction

Consider the following differential inclusion:

$$\begin{pmatrix} x'(t) \\ \varepsilon y'(t) \end{pmatrix} \in F\big(x(t), y(t)\big), \tag{1}$$

where ε is a small parameter, $(x, y) \in \mathbb{R}^M \times \mathbb{R}^N$, $t \geq 0$, and F is a set-valued map. We call S_ε the set of solutions of (1). The evolutions of the two state variables x and y of the system are of different scale. We call x the *slow variable* and y the *fast variable.* For $\varepsilon = 0$ we get the following equation:

$$\begin{pmatrix} x'(t) \\ 0 \end{pmatrix} \in F\big(x(t), y(t)\big), \tag{2}$$

*Laboratoire de Mathématiques, Unité CNRS UMR 6205, Université de Bretagne Occidentale, 6, Avenue Victor Le Gorgeu, BP 809, 29285 Brest, France, e-mail: frederique.watbled@univ-rennes1.fr

which is not a differential inclusion anymore. We call S_0 the set of solutions of (2), which consists of all the pairs (x, y) with x absolutely continuous, y measurable, satisfying (2) almost everywhere.

One important motivation is applications to singularly perturbed controlled systems:

$$\left\{\begin{array}{l} x'(t) = f\big(x(t), y(t), \alpha(t)\big) \\ \varepsilon y'(t) = g\big(x(t), y(t), \alpha(t)\big), \end{array}\right. \tag{3}$$

where $\alpha : [0, \infty[\to A$ is a measurable function with values in a compact space A. Indeed we can write (3) under the equivalent form (1) by setting:

$$F(x, y) = \{\big(f(x, y, a), g(x, y, a)\big);\ a \in A\}.$$

When F is single-valued, the inclusions (1) and (2) reduce respectively to:

$$\left\{\begin{array}{l} x'(t) = f\big(x(t), y(t)\big) \\ \varepsilon y'(t) = g\big(x(t), y(t)\big), \end{array}\right. \tag{4}$$

$$\left\{\begin{array}{l} x'(t) = f\big(x(t), y(t)\big) \\ \quad 0 = g\big(x(t), y(t)\big). \end{array}\right. \tag{5}$$

Assuming that equation (4) has a unique solution $(x_\varepsilon, y_\varepsilon)$ and equation (5) has a unique solution (x, y) when the initial condition is fixed, Tikhonov's theorem ([24], [25]) gives sufficient conditions to ensure continuity of the solution map $\varepsilon \to S_\varepsilon$ at $\varepsilon = 0$, in the sense that:

$$\begin{array}{l} \sup_{t\in[0,T]} |x_\varepsilon(t) - x(t)| \to_{\varepsilon\to 0} 0, \\ \sup_{t\in[t_1,T]} |y_\varepsilon(t) - y(t)| \to_{\varepsilon\to 0} 0 \text{ for any } t_1 > 0. \end{array}$$

Roughly speaking, this happens when the equation $g(x, y) = 0$ has an isolated solution $y = \varphi(x)$ which is stable, i.e. $\frac{\partial g}{\partial y}\big(x, \varphi(x)\big) < 0$, and when the initial condition (x_0, y_0) is such that y_0 belongs to the domain of attraction of the root $\varphi(x_0)$ of the equation $g(x_0, y) = 0$. As was shown by Hoppensteadt ([18]), a condition of stability on the degenerate equation:

$$x' = f\big(x, \varphi(x)\big),$$

implies uniform convergence of $(x_\varepsilon, y_\varepsilon)$ towards (x, y) on infinite intervals:

$$\begin{array}{l} \sup_{t\geq 0} |x_\varepsilon(t) - x(t)| \to_{\varepsilon\to 0} 0, \\ \sup_{t\geq t_1} |y_\varepsilon(t) - y(t)| \to_{\varepsilon\to 0} 0 \text{ for any } t_1 > 0. \end{array}$$

In this paper, we obtain a new result of Tikhonov-Hoppensteadt type. Indeed we prove that every sequence of solutions in S_ε converges to a solution in S_0: in other words, the set valued map $\varepsilon \to S_\varepsilon$ is upper semi-continuous at $\varepsilon = 0$. This question has been investigated in many papers for finite horizon, for instance in [7] and [14]. In [2], [28], [1], Artstein and Vigodner show the convergence of the slow trajectories x_ε towards a solution of a differential inclusion which depends on the invariant measures of the fast subsystems:

$$y' \in P_N F(x, y), \quad x \text{ fixed}, \tag{6}$$

where P_N is the projection onto $\mathbb{R}^N$. In [15], [16], [17] (see also [22]), Gaitsgory and Grammel show the convergence in the sense of Hausdorff's distance of the set of slow variables towards the set of solutions of a differential inclusion $x' \in G(x)$ obtained by an averaging method. The result holds under hypothesis of stability or controllability of the fast subsystems (see also [13]). In [20], [21], Quincampoix shows the existence of some convergent solutions $(x_\varepsilon, y_\varepsilon)$ towards some solution of (2), for maps $F = F_1 \times F_2$ and under an hypothesis of stability on the fast inclusion $y' \in F_2(y)$ (cf also [23]). In [27], Veliov provides a Tikhonov type theorem which asserts the upper semicontinuity of the solution map $\varepsilon \mapsto S_\varepsilon$ for the Tikhonov metric (the metric of uniform convergence on the compacts of $[0,T]\times]0,T]$) under a stability condition plus some structural assumption about F (see also [12], [26]).

In this paper we assume that there exists a Lipschitz set-valued map K from $\mathbb{R}^M$ to subsets of $\mathbb{R}^N$, with compact convex values, such that every set $K(x)$ is asymptotically stable with respect to the fast subsystem (6). We then define:

$$D(x) = \overline{co}\Big(P_M F\big(x, K(x)\big)\Big),$$

where $\overline{co}X$ stands for the closed convex hull of the set X, and P_M stands for the projection onto $\mathbb{R}^M$. We obtain the convergence of every sequence of slow states x_ε towards a solution of the differential inclusion $x' \in D(x)$ for the distance of uniform convergence on every compact of $[0, +\infty[$. Moreover, under an assumption of stability on the differential inclusion $x' \in D(x)$, we get the uniform convergence on the whole infinite interval $[0, +\infty[$. Adding a structural assumption about F we are able to obtain the convergence of solutions $(x_\varepsilon, y_\varepsilon) \in S_\varepsilon(x, y)$ towards a solution of S_0, for the distance of uniform convergence on every compact of $[0, +\infty[\times]0, +\infty[$.

To prove our results we construct Liapunov functions $W(x, y)$ associated to the systems (6). In [19], it is shown that if a set is asymptotically stable then there exists a smooth Liapunov function associated to it. This result is improved in [6]. Here we construct a Liapunov function W which has the remarkable property to be smooth with respect to both variables.

The paper is organized as follows: in the first part we give some definitions and assumptions. In part 2 we construct a Liapunov function $W(x, y)$ which is smooth in both variables. In part 3 we use this function to show the convergence of every sequence of $S_\varepsilon(x,y)$ towards an element of S_0.

1 Preliminaries and assumptions

Recall that a solution of (1) on an interval I is a couple $(x_\varepsilon, y_\varepsilon)$ of absolutely continuous functions such that:

$$\big(x_\varepsilon'(t), \varepsilon y_\varepsilon'(t)\big) \in F\big(x_\varepsilon(t), y_\varepsilon(t)\big) \text{ for almost every } t \in I.$$

From now on we make the following assumption:

(HF) *The map F has nonempty convex compact values, and is L-Lipschitz on $\mathbb{R}^M \times \mathbb{R}^N$. Then F has linear growth, i. e.: there exists $c > 0$ such that, for any $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$, we have:*

$$\sup_{(u,v)\in F(x,y)} (|u| + |v|) \leq c(|x| + |y| + 1).$$

Then we know that (1) has solutions defined on $[0, +\infty[$ (we refer to [4], [3], [5], [11], for the general theory of differential inclusions). We call $S_\varepsilon(x,y)$ the set of all the trajectories of (1) which start at (x,y):

$$S_\varepsilon(x,y) = \{\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big) \text{ solution of (1) and } x(0) = x,\ y(0) = y\}.$$

If we make the change of variables $\tau = \frac{t}{\varepsilon}$ in the system (1), and if we set $X(\tau) = x(\varepsilon\tau)$, we obtain:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} \in \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} F(X,Y).$$

When ε tends to 0, we are led to consider the following associated differential inclusion:

$$Y' \in P_N F(x,Y), \tag{7}$$

where x is fixed in $\mathbb{R}^M$, and P_N stands for the projection of $\mathbb{R}^M \times \mathbb{R}^N$ on $\mathbb{R}^N$. We call $S_x(y)$ the set of all the trajectories of (7) which start at y:

$$S_x(y) = \{y(\cdot),\ y'(s) \in P_N F\big(x, y(s)\big) a.e. \text{ and } y(0) = y\}.$$

For every $x \in \mathbb{R}^M$, we define $E(x)$ as the set of equilibrium points of (7):

$$E(x) = \{y \in \mathbb{R}^N;\ 0 \in P_N F(x,y)\}.$$

We assume that there exists a set-valued map K defined on $\mathbb{R}^M$ such that:

(H1) *The map K is Lipschitz on $\mathbb{R}^M$ with a Lipschitz constant L_K; for every $x \in \mathbb{R}^M$, the set $K(x)$ is a compact convex subset of $\mathbb{R}^N$.*

We then define a set-valued map D on $\mathbb{R}^M$ by setting:

$$D(x) = \overline{co}\Big(P_M F\big(x, K(x)\big)\Big),$$

where $\overline{co}X$ stands for the closed convex hull of the set X, and P_M stands for the projection onto $\mathbb{R}^M$. It is clear that D has non empty compact convex values and is Lipschitz, with a constant L_D less or equal than $L(1+L_K)$. We call $S_D(x)$ the set of trajectories of the differential inclusion $x' \in D(x)$ which start at x. We note B_r the open ball centered at the origin of radius r, and $\overline{B}_r$ the closed one, the ambient metric space is always clear from the context. If A is a subset of a normed space X, and $x \in X$, we denote by $d_A(x)$ the distance from x to A:

$$d_A(x) = \inf\{|x - z|,\ z \in A\}.$$

We recall the definition of uniform global asymptotic stability (UGAS) (we refer to [6], [11], [9], [10], [8], [19] for instance, for the definitions and properties of this notion). We say that a set A is UGAS for the differential inclusion $x' \in D(x)$ if for any initial condition x, every solution $x(\cdot)$ starting at x comes close to A when t goes to infinity, at a rate which depends only on the distance $d_A(x)$ from the initial position to A. More precisely we shall use the following definition:

Definition 1.1 *A set A is UGAS for the differential inclusion $x' \in D(x)$ if there exists two functions $m :]0,+\infty[\to]0,+\infty[$ and $T :]0,\infty[\times]0,+\infty[\to]0,+\infty[$, with m nondecreasing and $m(r)_{r\to 0} \to 0$, such that, for any $R > 0$, any $\nu > 0$, any $x \in A + \overline{B}_R$, and any trajectory $x(\cdot)$ starting at x, we have:*

1. $d_A\big(x(t)\big) \to_{t\to+\infty} 0$,
2. $d_A\big(x(t)\big) \leq m(R) \quad \forall t \geq 0$,
3. $d_A\big(x(t)\big) \leq \nu \quad \forall t \geq T(R,\nu)$.

Notice that if a set A is UGAS for D, then A is automatically invariant for D: if x is in A, every trajectory starting from x must remain in A.

We assume the following:

(H2) *There exists a nonempty compact subset A of $\mathbb{R}^M$ which is UGAS for the differential inclusion $x' \in D(x)$, with m and T as in definition 1.1.*

(H3) *Every set $K(x)$ is UGAS for the differential inclusion (7), with functions m_R and T_R as in definition 1.1, being such that: $\forall R > 0$, $\forall x \in \overline{B}_R$, $\forall r > 0$, $\forall y \in$*

$K(x) + \overline{B}_r$, $\forall \nu > 0$, *every trajectory* $y(\cdot) \in S_x(y)$ *satisfies:*

1. $d_{K(x)}\big(y(t)\big) \to_{t \to +\infty} 0$,
2. $d_{K(x)}\big(y(t)\big) \leq m_R(r) \quad \forall t \geq 0$,
3. $d_{K(x)}\big(y(t)\big) \leq \nu \quad \forall t \geq T_R(r, \nu)$.

Notice that the map K is the multivalued analog of the implicit function $y = \varphi(x)$ defined by the equation $g(x, y) = 0$ in the Tikhonov approach, and the map D is the analog of the function $d(x) = f\big(x, \varphi(x)\big)$. One difference is that $K(x)$ is not assumed to be contained in $E(x)$. Yet it is invariant since it is UGAS, hence the Equilibrium Theorem (Th. 3.2.1 in [5]) ensures that $K(x) \cap E(x) \neq \emptyset$.

2 Construction of Lyapunov functions

In [6], it is shown (Th.4.5) that if the origin is UGAS for the differential inclusion $x' \in D(x)$, then, for any $\lambda > 0$, there exists a Lyapunov function $V : \mathbb{R}^M \to \mathbb{R}^+$ of class C^∞ and two functions a, b in $\mathcal{K}_0^\infty$ such that:

$$\begin{aligned} &a(|x|) \leq V(x) \leq b(|x|) \text{ for all } x, \\ &\langle \nabla V(x), v \rangle \leq -\lambda V(x) \text{ for all } v \in D(x). \end{aligned}$$

Here $\mathcal{K}_0^\infty$ is the standard notation for the class of functions $\alpha : [0, +\infty[\to [0, +\infty[$ which are continuous, strictly increasing, with $\alpha(0) = 0$ and $\lim_{r \to +\infty} \alpha(r) = +\infty$. By adapting the proof to the case where a set A is UGAS instead of the single point 0, we get the following:

PROPOSITION 2.1 *Let assumption* **(H2)** *hold, let* $\lambda > 0$. *Then there exists a function* $V : \mathbb{R}^M \to \mathbb{R}^+$ *of class* C^∞ *and two functions* a_V, b_V *in* $\mathcal{K}_0^\infty$ *such that, for every* $x \in \mathbb{R}^M$, *we have:*

$$\begin{aligned} &a_V\big(d_A(x)\big) \leq V(x) \leq b_V\big(d_A(x)\big), \\ &\langle \nabla V(x), v \rangle \leq -\lambda V(x) \text{ for all } v \in D(x). \end{aligned}$$

REMARK 2.1 Notice that the last inequality implies that every trajectory $x(\cdot)$ of $x' \in D(x)$ satisfies:

$$\forall t \geq 0, \; V\big(x(t)\big) \leq e^{-\lambda t} V\big(x(0)\big),$$

because for almost every $t \geq 0$,

$$\frac{d}{dt}\Big(V\big(x(t)\big)e^{\lambda t}\Big) = \Big(\langle \nabla V\big(x(t)\big), x'(t) \rangle + \lambda V\big(x(t)\big)\Big)e^{\lambda t} \leq 0.$$

Proposition 2.1 applied to the maps $P_N F(x,\cdot)$ and the sets $K(x)$ when hypothesis **(H3)** holds proves the existence of a Lyapunov function $W(x,y)$. The remarkable fact is that W is smooth in *both variables.* Actually we prove directly the proposition for W, and consider Proposition 2.1 as a consequence. We do not give too much details, because the procedure is the same as in [6] (see also [19]).

PROPOSITION 2.2 *Under the hypothesis* **(HF)**, **(H1)** *and* **(H3)**, *there exists a function* $W : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^+$ *of class* C^∞, *a function* a_W *in* $\mathcal{K}_0^\infty$, *and a family of functions* $(b_R)_{R>0}$ *in* $\mathcal{K}_0^\infty$, *such that: for any* $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$, *we have:*

$$\begin{array}{l} a_W\big(d_{K(x)}(y)\big) \le W(x,y), \\ |x| \le R \Rightarrow W(x,y) \le b_R\big(d_{K(x)}(y)\big), \\ \langle \nabla_y W(x,y), v\rangle \le -\lambda W(x,y) \text{ for all } v \in P_N F(x,y). \end{array}$$

REMARK 2.2 Notice here again that the last inequality implies that every trajectory $y(\cdot)$ of $y' \in P_N F(x,y)$ satisfies:

$$\forall t \ge 0,\ W\big(x,y(t)\big) \le e^{-\lambda t} W\big(x,y(0)\big).$$

Proof. Without loss of generality, we assume that the functions m_R given in **(H3)** are in $\mathcal{K}_0^\infty$, and that $T_R(r,\nu)$ is monotone in each of the variables R, r, and ν (nondecreasing in R, r, nonincreasing in ν), when the two others are fixed.

First step: For $q \in \mathbb{N}^\star$, $r \ge 0$, and $x \in \mathbb{R}^M$, we set:

$$\begin{array}{l} G_q(r) = \max(0, r - \frac{1}{q}), \\ W_q(x,y) = \sup_{\varphi \in S_x(y)} \sup_{t \ge 0} e^{2\lambda t} G_q\big(d_{K(x)}(\varphi(t))\big). \end{array}$$

We deduce from **(H3)** that $\forall x \in \overline{B}_R$, $\forall y \in K(x) + \overline{B}_r$, we have:

$$W_q(x,y) = \sup_{\varphi \in S_x(y)} \sup_{t \in [0, T_R(r,\frac{1}{q})]} e^{2\lambda t} G_q\big(d_{K(x)}(\varphi(t))\big) \le e^{2\lambda T_R(r,\frac{1}{q})} m_R(r).$$

It is easy to see that the supremum is attained and that $W_q(x,y)$ is locally Lipschitz on $\mathbb{R}^M \times \mathbb{R}^N$:

LEMMA 2.3 *Let* $R > 0$, $r > 0$. *Let* $\delta = \max_{|x| \le R, |y| \le r} d_{K(x)}(y)$, $T = T_R(\delta, \frac{1}{q})$. *Then, for all* (x_1,y_1), $(x_2,y_2) \in \overline{B}_R \times \overline{B}_r$, *we have:*

$$|W_q(x_1,y_1) - W_q(x_2,y_2)| \le C_R(r,q)(|x_1 - x_2| + |y_1 - y_2|),$$

where $C_R(r,q) = e^{2\lambda T}(e^{LT} + L_K)$.

Second step: For $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$, we set:

$$W_L(x,y) = \sum_{q=1}^{+\infty} \frac{2^{-q}}{1+C_q(q,q)} e^{-2\lambda T_q(q,\frac{1}{q})} W_q(x,y),$$

and we define, for $r \geq 0$:

$$a_L(r) = \sum_{q=1}^{+\infty} \frac{2^{-q}}{1+C_q(q,q)} e^{-2\lambda T_q(q,\frac{1}{q})} G_q(r).$$

The function a_L is in $\mathcal{K}_0^\infty$. Eventually we choose $b_{R,L}$ in $\mathcal{K}_0^\infty$ such that $\tilde{m}_R(r) \leq b_{R,L}(r)$ for every $r \geq 0$, where $\tilde{m}_R$ is defined by:

$$\tilde{m}_R(r) = \Big[\sum_{q \leq \max(r,R)} \frac{2^{-q}}{1+C_q(q,q)} e^{2\lambda\left(T_R(r,\frac{1}{q}) - T_q(q,\frac{1}{q})\right)} + 1 \Big] m_R(r).$$

Then W_L satisfies the following:

LEMMA 2.4 *For any $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$, we have:*

$$a_L\big(d_{K(x)}(y)\big) \leq W_L(x,y).$$

For any $(x,y) \in \overline{B}_R \times \mathbb{R}^N$, we have:

$$W_L(x,y) \leq b_{R,L}\big(d_{K(x)}(y)\big).$$

The function W_L is 1-Lipschitz on $\mathbb{R}^M \times \mathbb{R}^N$, and it is non increasing along the trajectories of $y' \in P_N F(x,y)$:

$$\forall \varphi \in S_x(y),\ \forall h \geq 0,\ W_L\big(x, \varphi(h)\big) \leq e^{-2\lambda h} W_L(x,y).$$

The function W_L is differentiable in almost every $(x,y) \in \mathbb{R}^M \times \mathbb{R}^N$, and every such (x,y) satisfies:

$$\forall v \in P_N F(x,y),\ \langle \nabla_y W_L(x,y), v \rangle \leq -2\lambda W_L(x,y).$$

Third step: We regularize W_L by convolution. Let Ω the open set defined by:

$$\Omega = \{(x,y) \in \mathbb{R}^M \times \mathbb{R}^N,\ y \notin K(x)\}.$$

LEMMA 2.5 *Let S be a compact subset of Ω, let $\varepsilon > 0$. Then there exists $\overline{W}$ in $C_c^\infty(\mathbb{R}^M \times \mathbb{R}^N)$ with supp$\overline{W} \subset \Omega$ such that:*

$\|\overline{W} - W_L\|_{L^\infty(S)} < \varepsilon$,
$\langle \nabla_y \overline{W}(x,y), v \rangle \leq -\frac{3}{2}\lambda W_L(x,y)$ *for any $(x,y) \in S$ and any $v \in P_N F(x,y)$.*

Then we take (O_n) a locally finite covering of Ω by a family of relatively compact open sets, and (ϕ_n) a partition of unity C^∞ associated to (O_n), that is:

$$\begin{array}{l} \phi_n \in C_c^\infty(\mathbb{R}^M \times \mathbb{R}^N),\ 0 \leq \phi_n \leq 1, \\ S_n = supp\ \phi_n \subset O_n\ \forall n, \\ \sum_{n\in\mathbb{N}^\star} \phi_n(x) = 1\ \forall x \in \Omega. \end{array}$$

We define:

$$\begin{array}{l} q_n = \sup_{(x,y)\in S_n} \sup_{v\in P_N F(x,y)} |\langle \nabla_y \phi_n(x,y), v\rangle|, \\ \varepsilon_n = \frac{2^{-n-2}}{1+q_n}\frac{\lambda}{\lambda+1} \min_{S_n} W_L, \end{array}$$

and we apply the preceding lemma to every compact S_n, with $\varepsilon_n > 0$, to obtain a function W_n. Eventually we set:

$$\begin{array}{l} W(x,y) = \sum_{n=1}^{+\infty} \phi_n(x,y) W_n(x,y) \text{ for } (x,y) \in \mathbb{R}^M \times \mathbb{R}^N, \\ a_W(r) = \frac{3}{4} a_L(r), \\ b_R(r) = \frac{5}{4} b_{R,L}(r), \end{array}$$

and we check that W has the claimed properties. ■

3 Convergence of trajectories

We are now ready to state the main result of the paper:

THEOREM 3.1 *Let assumptions* **(HF)**, **(H1)**, **(H2)**, **(H3)** *hold. Then, for any* $x \in \mathbb{R}^M$, *there exists* $r > 0$ *such that, for any* $y \in K(x) + B_r$, *and for every sequence* $(x_\varepsilon, y_\varepsilon) \in S_\varepsilon(x,y)$, *there exists* $\overline{x} \in S_D(x)$ *such that:*

$$\forall t_1 > 0,\ \sup_{t\geq t_1} d_{K(x_\varepsilon(t))}\big(y_\varepsilon(t)\big) \rightarrow_{\varepsilon\to 0} 0, \tag{8}$$

$$\forall T > 0,\ \sup_{t\in[0,T]} |x_\varepsilon(t) - \overline{x}(t)| \rightarrow_{\varepsilon\to 0} 0. \tag{9}$$

Moreover $\forall \eta > 0$, $\exists T_\eta > 0$, *such that* $\forall T > T_\eta$, $\exists\ \varepsilon_0$ *such that, for all* $\varepsilon \in]0, \varepsilon_0[$, *we have:*

$$\sup_{t\in[0,T]} |x_\varepsilon(t) - \overline{x}(t)| + \sup_{t\geq T} d_A\big(x_\varepsilon(t)\big) + \sup_{t\geq T} d_A\big(\overline{x}(t)\big) \leq \eta. \tag{10}$$

Proof. Without loss of generality, we can assume that the maps m_R and m given by the hypothesis are in $\mathcal{K}_0^\infty$, and that T and T_R are monotone in both variables. We choose V, a_V, and b_V as in Proposition 2.1, and W, a_W, b_R as in Proposition 2.2. For $(c,d) \in \mathbb{R}^+ \times \mathbb{R}^+$, we define the level set:

$$\Gamma_{c,d} = \{(x,y) \in \mathbb{R}^M \times \mathbb{R}^N;\ V(x) \leq c \text{ and } W(x,y) \leq d\}.$$

Notice that $\Gamma_{c,d}$ is compact because:

$$V(x) \leq c \Rightarrow x \in A + \overline{B}_{a_V^{-1}(c)},$$

$$W(x,y) \leq d \Rightarrow y \in K(0) + \overline{B}_{L_K|x| + a_W^{-1}(d)}.$$

In the following lemma we prove that for d small enough compared to c, the set $\Gamma_{c,d}$ is invariant for (1) for ε small enough.

LEMMA 3.2 *For all $c > 0$, there exists $d(c) > 0$ such that, for all $d \in [0, d(c)]$, there exists $\varepsilon(c,d) > 0$ such that $\Gamma_{c,d}$ is invariant for (1) for any $\varepsilon \in]0, \varepsilon(c,d)[$.*

Proof. It suffices to prove that any trajectory starting from the boundary $\partial\Gamma_{c,d}$ stays inside $\Gamma_{c,d}$. So let $(x,y) \in \partial\Gamma_{c,d}$, and $(x_\varepsilon, y_\varepsilon)$ be a trajectory in $S_\varepsilon(x,y)$. Then $V(x) = c$, or $W(x,y) = d$. For almost every t, we have:

$$\begin{aligned}\tfrac{d}{dt}V\big(x_\varepsilon(t)\big) &= \nabla V\big(x_\varepsilon(t)\big) \cdot x'_\varepsilon(t)\\ &= \nabla V\big(x_\varepsilon(t)\big) \cdot \big(x'_\varepsilon(t) - d_\varepsilon(t)\big) + \nabla V\big(x_\varepsilon(t)\big) \cdot d_\varepsilon(t),\end{aligned}$$

with $d_\varepsilon(t) \in D\big(x_\varepsilon(t)\big)$ such that:

$$|x'_\varepsilon(t) - d_\varepsilon(t)| = dist\,\Big(x'_\varepsilon(t), D\big(x_\varepsilon(t)\big)\Big),$$

from which we deduce that:

$$\begin{aligned}\tfrac{d}{dt}V\big(x_\varepsilon(t)\big) &\leq |\nabla V\big(x_\varepsilon(t)\big)|dist\,\Big(x'_\varepsilon(t), D\big(x_\varepsilon(t)\big)\Big) - \lambda V\big(x_\varepsilon(t)\big)\\ &\leq |\nabla V\big(x_\varepsilon(t)\big)|dist\,\Big(x'_\varepsilon(t), P_M F\big(x_\varepsilon(t), K(x_\varepsilon(t))\big)\Big) - \lambda V\big(x_\varepsilon(t)\big)\\ &\leq |\nabla V\big(x_\varepsilon(t)\big)|L d_{K\left(x_\varepsilon(t)\right)}\Big(y_\varepsilon(t)\Big) - \lambda V\big(x_\varepsilon(t)\big) := \alpha(t).\end{aligned}$$

If $V(x) = c$ then:

$$\alpha(0) = |\nabla V(x)| L d_{K(x)}(y) - \lambda c \leq -\lambda c + \rho L a_W^{-1}(d),$$

where ρ is a constant depending only on c. We choose $d(c) > 0$ such that $a_W^{-1}\big(d(c)\big) < \frac{\lambda c}{L\rho}$. Then for any $d \in [0, d(c)]$, we get:

$$\frac{d}{dt}V\big(x_\varepsilon(t)\big) \leq \alpha(t), \quad \alpha(0) < 0,$$

where α is continuous, hence we deduce that $V\big(x_\varepsilon(t)\big) < V(x) = c$ for small $t > 0$.

Now let us fix $d \in]0, d(c)]$. We have, for almost every t:

$$\begin{aligned}&\tfrac{d}{dt}W\big(x_\varepsilon(t), y_\varepsilon(t)\big) = \nabla_x W\big(x_\varepsilon(t), y_\varepsilon(t)\big) \cdot x'_\varepsilon(t) + \nabla_y W\big(x_\varepsilon(t), y_\varepsilon(t)\big) \cdot y'_\varepsilon(t)\\ &\leq |\nabla_x W\big(x_\varepsilon(t), y_\varepsilon(t)\big)| c\big(|x_\varepsilon(t)| + |y_\varepsilon(t)| + 1\big) - \tfrac{\lambda}{\varepsilon} W\big(x_\varepsilon(t), y_\varepsilon(t)\big) := \beta(t),\end{aligned}$$

because $\varepsilon y_\varepsilon'(t)$ belongs to $P_N F\big(x_\varepsilon(t), y_\varepsilon(t)\big)$. If $W(x,y) = d$ then:

$$\beta(0) \leq \rho - \frac{\lambda d}{\varepsilon},$$

where ρ is a constant depending only on c and d. We choose $\varepsilon(c,d) > 0$ such that $\varepsilon(c,d) < \frac{\lambda d}{\rho}$, so that for any $\varepsilon \in]0, \varepsilon(c,d)[$, we have:

$$\frac{d}{dt} W\big(x_\varepsilon(t), y_\varepsilon(t)\big) \leq \beta(t), \quad \beta(0) < 0,$$

where β is continuous, hence we deduce that $W\big(x_\varepsilon(t), y_\varepsilon(t)\big) < W(x,y) = d$ for small $t > 0$. ■

Now let $x \in \mathbb{R}^M$, let $R > |x|$, and let $c_1 = b_V(R)$. We can assume without loss of generality that 0 belongs to A, so we have $R > d_A(x)$ hence $V(x) < c_1$. Let $d_1 = d(c_1)$ be given by Lemma 3.2, and define r by:

$$r = b_R^{-1}(d_1).$$

Let y be in $K(x) + B_r$, and let $(x_\varepsilon, y_\varepsilon)$ be a sequence of elements in $S_\varepsilon(x,y)$ with $\varepsilon \in]0, \varepsilon(c_1, d_1)[$. As $(x,y) \in \Gamma(c_1, d_1)$ we know from Lemma 3.2 that $(x_\varepsilon, y_\varepsilon)$ remains in $\Gamma(c_1, d_1)$. As a consequence we get that $(x_\varepsilon, y_\varepsilon)$ is uniformly bounded:

$$\forall t \geq 0, \ |x_\varepsilon(t)| + |y_\varepsilon(t)| \leq \rho,$$

where ρ is a constant depending only on R. In the following the value of ρ may vary, but it always depends only on R.

LEMMA 3.3 *For every $\gamma > 0$, there exists $\tau_\gamma > 0$ and $\varepsilon_\gamma > 0$ such that, for every $\varepsilon \in]0, \varepsilon_\gamma[$, we have:*

$$\forall t \geq \varepsilon \tau_\gamma, \ d_{K(x_\varepsilon(t))}\big(y_\varepsilon(t)\big) \leq \gamma, \tag{11}$$

$$\forall T > 0, \ \int_0^T d_{K\left(x_\varepsilon(s)\right)}\big(y_\varepsilon(s)\big) \ ds \leq \gamma(1+T). \tag{12}$$

Proof. Let $\gamma > 0$ be such that $a_W(\gamma) < d_1$. We make the change of variable $s = \frac{t}{\varepsilon}$, and set:

$$X_\varepsilon(s) = x_\varepsilon(\varepsilon s),$$
$$Y_\varepsilon(s) = y_\varepsilon(\varepsilon s).$$

Then $Y_\varepsilon'(s) \in P_N F\big(X_\varepsilon(s), Y_\varepsilon(s)\big)$ for almost every s, hence:

$$\begin{aligned} dist\left(\begin{pmatrix} X_\varepsilon'(s) \\ Y_\varepsilon'(s) \end{pmatrix}, \{0\} \times P_N F\big(x, Y_\varepsilon(s)\big) \right) &\leq |X_\varepsilon'(s)| + L|X_\varepsilon(s) - x| \\ &\leq (1 + Ls) \sup_{u \in [0,s]} |X_\varepsilon'(u)| \\ &\leq \varepsilon\rho(1 + Ls), \end{aligned}$$

because F is of linear growth and $(x_\varepsilon, y_\varepsilon)$ is bounded. Applying Filippov's theorem we choose a solution $\big(X(\cdot), Y(\cdot)\big)$ of the differential inclusion

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} \in \{0\} \times P_N F(x, Y)$$

such that $X(s) = x$ for all s, $Y(0) = y$, and

$$\forall s \geq 0,\ |x - X_\varepsilon(s)| + |Y(s) - Y_\varepsilon(s)| \leq e^{Ls}\varepsilon\rho.$$

We get:

$$\begin{aligned} d_{K\left(X_\varepsilon(s)\right)}\big(Y_\varepsilon(s)\big) &\leq d_{K(x)}\big(Y(s)\big) + L_K|x - X_\varepsilon(s)| + |Y(s) - Y_\varepsilon(s)| \\ &\leq d_{K(x)}\big(Y(s)\big) + (1 + L_K)e^{Ls}\varepsilon\rho. \end{aligned} \tag{13}$$

Now $Y(\cdot)$ is a trajectory of (7), hence satisfies:

$$W\big(x, Y(s)\big) \leq e^{-\lambda s} W(x, y) \leq e^{-\lambda s} d_1,$$

from which we deduce:

$$d_{K(x)}\big(Y(s)\big) \leq a_W^{-1}(e^{-\lambda s} d_1).$$

Replacing in (13), we get:

$$d_{K\left(X_\varepsilon(s)\right)}\big(Y_\varepsilon(s)\big) \leq (1 + L_K)e^{Ls}\varepsilon\rho + a_W^{-1}(e^{-\lambda s} d_1). \tag{14}$$

Now we choose $\tau_\gamma > 0$ such that:

$$a_W^{-1}(e^{-\lambda\tau_\gamma} d_1) \leq \frac{1}{2} b_\rho^{-1}(a_W(\gamma)), \tag{15}$$

and $\varepsilon_\gamma \in]0, \varepsilon\big(c_1, a_W(\gamma)\big)[$ such that:

$$(1 + L_K)e^{L\tau_\gamma}\varepsilon_\gamma\rho < \frac{1}{2} b_\rho^{-1}\big(a_W(\gamma)\big), \tag{16}$$

$$\varepsilon_\gamma \tau_\gamma a_W^{-1}(d_1) < \gamma. \tag{17}$$

Putting together (14), (15), and (16), we get, for any $\varepsilon \in]0, \varepsilon_\gamma[$:

$$d_{K\left(x_\varepsilon(\varepsilon\tau_\gamma)\right)}\big(y_\varepsilon(\varepsilon\tau_\gamma)\big) \leq b_\rho^{-1}(a_W(\gamma)),\ \text{hence } W\big(x_\varepsilon(\varepsilon\tau_\gamma), y_\varepsilon(\varepsilon\tau_\gamma)\big) \leq a_W(\gamma).$$

As we have chosen ε_γ smaller than $\varepsilon(c_1, a_W(\gamma))$, the trajectory remains inside $\Gamma_{c_1, a_W(\gamma)}$ forever, so we get:

$$\sup_{t \geq \varepsilon\tau_\gamma} d_{K\left(x_\varepsilon(t)\right)}\big(y_\varepsilon(t)\big) \leq \gamma, \tag{18}$$

which proves (11). Then from (17) and (18) we deduce that for any $T > 0$ and any $\varepsilon \in]0, \varepsilon_\gamma[$, we have:

$$\begin{aligned} \int_0^T d_{K\left(x_\varepsilon(s)\right)}\big(y_\varepsilon(s)\big)\ ds &= \int_0^{\varepsilon\tau_\gamma} d_{K\left(x_\varepsilon(s)\right)}\big(y_\varepsilon(s)\big)\ ds + \int_{\varepsilon\tau_\gamma}^T d_{K\left(x_\varepsilon(s)\right)}\big(y_\varepsilon(s)\big)\ ds \\ &\leq \varepsilon\tau_\gamma a_W^{-1}(d_1) + T\gamma \leq \gamma(1+T), \end{aligned}$$

which proves inequality (12). ■

Now let $t_1 > 0$, $\gamma > 0$. Let τ_γ and ε_γ be given by Lemma 3.3, where we choose ε_γ smaller than $\frac{t_1}{\tau_\gamma}$. Then, for every $\varepsilon \in]0, \varepsilon_\gamma[$, we get:

$$\sup_{t \geq t_1} d_{K(x_\varepsilon(t))}\big(y_\varepsilon(t)\big) \leq \sup_{t \geq \varepsilon\tau_\gamma} d_{K(x_\varepsilon(t))}\big(y_\varepsilon(t)\big) \leq \gamma,$$

which gives (8).

On the other hand, we have, for almost every t:

$$\begin{aligned} dist\big(x'_\varepsilon(t), D(x_\varepsilon(t))\big) &\leq dist\Big(x'_\varepsilon(t), P_M\big(x_\varepsilon(t), K(x_\varepsilon(t))\big)\Big) \\ &\leq L d_{K(x_\varepsilon(t))}\big(y_\varepsilon(t)\big). \end{aligned}$$

Applying Filippov's theorem, we can find a solution $\overline{x}_\varepsilon$ of $S_D(x)$ such that:

$$\forall t \geq 0,\ |x_\varepsilon(t) - \overline{x}_\varepsilon(t)| \leq L e^{tL_D} \int_0^t d_{K(x_\varepsilon)(s)}\big(y_\varepsilon(s)\big)\ ds. \tag{19}$$

There exists a subsequence, which we still call $\overline{x}_\varepsilon$, which converges uniformly on every compact towards a function $\overline{x} \in S_D(x)$. Let $T > 0$, let $\gamma > 0$, and choose ε_γ as before and also such that:

$$\forall \varepsilon \in]0, \varepsilon_\gamma[, \quad \sup_{t \in [0,T]} |\overline{x}(t) - \overline{x}_\varepsilon(t)| < \gamma. \tag{20}$$

Putting together (19), (20), and (12), we get:

$$\forall \varepsilon \in]0, \varepsilon_\gamma[, \quad \sup_{t \in [0,T]} |\overline{x}(t) - x_\varepsilon(t)| < L e^{TL_D}(1+T)\gamma + \gamma,$$

from which we deduce (9).

Now we fix $\eta > 0$, $\delta = \frac{\eta}{3}$, $c = a_V(\delta)$, $d = d(c)$ as in Lemma 3.2. We choose $T_\eta > T(R, \nu)$, where ν is defined by:

$$\nu = \min\big(\delta, \frac{1}{2} b_V^{-1}(c)\big).$$

We also ask to T_η to be greater than $\varepsilon_\gamma \tau_\gamma$, where ε_γ and τ_γ are given by Lemma 3.3 when applied to $\gamma = b_\rho^{-1}(d)$. Then:

$$\forall t \geq T_\eta,\ d_A\big(\overline{x}(t)\big) < \nu. \tag{21}$$

Let $T > T_\eta$. Then:

$$d_A\big(x_\varepsilon(T)\big) \leq d_A\big(\overline{x}(T)\big) + \sup_{t\in[0,T]} |x_\varepsilon(t) - \overline{x}(t)|. \tag{22}$$

Let $\varepsilon_0 \in]0, \varepsilon_\gamma[$ be such that, for all $\varepsilon \in]0, \varepsilon_0[$, we have:

$$\sup_{t\in[0,T]} |x_\varepsilon(t) - \overline{x}(t)| < \nu. \tag{23}$$

Then putting together (21), (22), and (23), we get:

$$d_A\big(x_\varepsilon(T)\big) \leq b_V^{-1}(c), \text{ hence } V\big(x_\varepsilon(T)\big) \leq c.$$

From the inequality:

$$\sup_{t\geq T} W\big(x_\varepsilon(t), y_\varepsilon(t)\big) \leq d \quad \forall \varepsilon \in]0, \varepsilon_0[,$$

which comes from (11), we deduce thanks to Lemma 3.2 that:

$$V\big(x_\varepsilon(t)\big) \leq c \text{ for all } t \geq T, \text{ hence } d_A\big(x_\varepsilon(t)\big) \leq a_V^{-1}(c) = \delta. \tag{24}$$

Putting together (21), (23), and (24), we get:

$$\sup_{t\in[0,T]} |x_\varepsilon(t) - \overline{x}(t)| + \sup_{t\geq T} d_A\big(x_\varepsilon(t)\big) + \sup_{t\geq T} d_A\big(\overline{x}(t)\big) \leq \eta,$$

so (10) is proved. ■

If A is a single point we get the immediate following corollary:

Corollary 3.4 *If A is a singleton then:*

$$\sup_{t\geq 0} |x_\varepsilon(t) - \overline{x}(t)| \to_{\varepsilon\to 0} 0.$$

If K is single-valued, we get as a corollary a Tikhonov type result:

Corollary 3.5 *If K is single-valued with $K(x) = \{k(x)\}$ for $x \in \mathbb{R}^M$, then:*

$\big(\overline{x}, k(\overline{x})\big) \in S_0$,
$(x_\varepsilon, y_\varepsilon) \to_{\varepsilon\to 0} \big(\overline{x}, k(\overline{x})\big)$ *uniformly on every compact of* $[0, +\infty[\times]0, +\infty[$.

If moreover the set A is reduced to a single point, then the convergence of $(x_\varepsilon, y_\varepsilon)$ towards $\big(\overline{x}, k(\overline{x})\big)$ is uniform on $[0, +\infty[\times[t_1, +\infty[$ for any $t_1 > 0$.

Proof. From Theorem 3.1 it is clear that $(x_\varepsilon, y_\varepsilon)$ converges uniformly to $\big(\overline{x}, k(\overline{x})\big)$ on every compact of $[0,+\infty[\times]0,+\infty[$, and that this convergence is uniform on $[0,+\infty[\times[t_1,+\infty[$ for any $t_1 > 0$ in case A is a singleton. The fact that $\big(\overline{x}, k(\overline{x})\big)$ is in S_0 is a consequence of the following lemma:

LEMMA 3.6 *Let $(x_\varepsilon, y_\varepsilon) \in S_\varepsilon(x,y)$ be such that $(x_\varepsilon, y_\varepsilon)$ converges uniformly towards $\big(\overline{x}, k(\overline{x})\big)$ on every compact of $[0,+\infty[\times]0,+\infty[$. Then $(\overline{x}, \overline{y})$ is in S_0.*

Proof: Let $[a,b] \subset]0,+\infty[$ with $0 < a < b$. The couple $(x_\varepsilon, y_\varepsilon)$ is bounded in $L^\infty(a,b)$, hence $(x'_\varepsilon, \varepsilon y'_\varepsilon)$ also, by linear growth of F. Hence there exists a subsequence $(x'_{\varepsilon_k}, \varepsilon_k y'_{\varepsilon_k})$ which converges to a limit (α, β) for the weak star topology $\sigma(L^\infty, L^1)$. As $L^\infty(a,b) \subset L^1(a,b)$, we have also convergence for the weak topology $\sigma(L^1, L^\infty)$. Then we deduce that:

$$x_{\varepsilon_k}(t) = x_{\varepsilon_k}(a) + \int_a^t x'_{\varepsilon_k}(s)\ ds \to \overline{x}(a) + \int_a^t \alpha(s)\ ds,$$

hence $\overline{x}'(t) = \alpha(t)$ for almost every t of $[a,b]$. In the same manner, $\beta(t) = 0$ for almost every t of $[a,b]$. As $\big((x_\varepsilon, y_\varepsilon), (x'_\varepsilon, \varepsilon y'_\varepsilon)\big)$ belongs to Graph$F$, we apply a classical convergence result (Theorem 2.4.4 in [3]) to deduce that $\big(\overline{x}'(t), 0\big)$ belongs to $F\big(\overline{x}(t), \overline{y}(t)\big)$ for almost every t of $[a,b]$. As this is true for any $0 < a < b$, we deduce that it is true for almost every $t \geq 0$. Therefore $(\overline{x}, \overline{y})$ belongs to S_0. ■

If the map K is not single-valued, we need some more hypothesis to obtain the convergence of the fast variable. We say that we have property **(HS)** if we have the following:

$$\forall x \in \mathbb{R}^M,\ \forall y \in K(x),\ P_M F(x,y) \times \{0\} \subset F(x,y).$$

For instance if $K(x) \subset E(x)$ and F is of the form $F_1 \times F_2$ (which means that the controls influencing the slow and the fast equations are independant), then property **(HS)** holds true. Notice that in general property **(HS)** implies that $K(x)$ is a subset of $E(x)$ for every $x \in \mathbb{R}^M$.

THEOREM 3.7 *Let assumptions* **(HF)**, **(H1)**, **(H2)**, **(H3)** *and* **(HS)** *hold. Then, for any $x \in \mathbb{R}^M$, there exists $r > 0$ such that, for any $y \in K(x) + B_r$, and for every sequence $(x_n, y_n) \in S_{\varepsilon_n}(x,y)$ with $\varepsilon_n \to 0$, we have:*

$$\forall t_1 > 0,\ \forall T > t_1,\ \delta_{[t_1,T]}\big((x_n,y_n), S_0\big) \to_{n\to+\infty} 0,$$

where the distance $\delta_{[t_1,T]}$ is defined by:

$$\delta_{[t_1,T]}\big((x_n,y_n),(x,y)\big) = \sup_{t\in[0,T]} |x_n(t) - x(t)| + \sup_{t\in[t_1,T]} |y_n(t) - y(t)|.$$

Proof. Let $x \in \mathbb{R}^M$, let $r > 0$ be given by Theorem 3.1, let $y \in K(x) + B_r$, and let (x_n, y_n) be a sequence in $S_{\varepsilon_n}(x, y)$ with $\varepsilon_n \to 0$. Fix $t_1 > 0$ and $T > t_1$. The result is then an immediate consequence of the following lemma:

LEMMA 3.8 *There exists a sequence $(\overline{x}_n, \overline{y}_n)$ such that:*

$$\begin{array}{l} \overline{x}_n' \in P_M F(\overline{x}_n, \overline{y}_n), \\ \overline{y}_n \in K(\overline{x}_n), \\ \delta_{[t_1,T]}\big((x_n, y_n), (\overline{x}_n, \overline{y}_n)\big) \to_{n\to+\infty} 0. \end{array}$$

Indeed, property **(HS)** then implies that $(\overline{x}_n', 0)$ belongs to $F(\overline{x}_n, \overline{y}_n)$, hence $(\overline{x}_n, \overline{y}_n)$ is solution of (2), hence it belongs to S_0.

Proof of the lemma: The idea of the proof is taken from [27]: for every n fixed, we construct a sequence of couples $\big((x_n^k, y_n^k)\big)_{k\in\mathbb{N}}$ which will converge towards the wanted $(\overline{x}_n, \overline{y}_n)$.

We set $x_n^0 = x_n$, $y_n^0 = y_n$.

<u>Step $k = 1$</u>: for every $t \in [0, T]$, we choose $y_n^1(t) \in K\big(x_n^0(t)\big)$ such that:

$$d_{K\left(x_n^0(t)\right)}\big(y_n^0(t)\big) = |y_n^0(t) - y_n^1(t)|.$$

Then ${x_n^0}' \in P_M F(x_n^0, y_n^0) \subset P_M F(x_n^0, y_n^1) + L|y_n^0 - y_n^1|\overline{B}$, hence by Filippov's theorem there exists x_n^1 such that:

$$\begin{array}{l} {x_n^1}' \in P_M F(x_n^1, y_n^1), \\ x_n^1(0) = x, \\ |x_n^0(t) - x_n^1(t)| \le Le^{Lt} \int_0^t d_{K\left(x_n(s)\right)}\big(y_n(s)\big)\, ds. \end{array}$$

We set:

$$\begin{array}{l} \beta_n = \sup_{t\in[0,T]} |x_n^0(t) - x_n^1(t)|, \\ a_n^k(t) = (LL_K e^{LT})^{k-1} \frac{t^{k-1}}{(k-1)!}\beta_n. \end{array}$$

<u>Step k</u>: We suppose constructed the first k couples verifying the following:

$$ {x_n^k}' \in P_M F(x_n^k, y_n^k), \tag{25}$$

$$ y_n^k(t) \in K\big(x_n^{k-1}(t)\big), \tag{26}$$

$$ |x_n^k(t) - x_n^{k-1}(t)| \le a_n^k(t),$$

$$ |y_n^k(t) - y_n^{k-1}(t)| \le L_K a_n^{k-1}(t) \text{ for } k \ge 2.$$

<u>Step $k + 1$</u>: As

$$y_n^k(t) \in K\big(x_n^{k-1}(t)\big) \subset K\big(x_n^k(t)\big) + L_K a_n^k(t)\overline{B},$$

we can choose $y_n^{k+1}(t) \in K\big(x_n^k(t)\big)$ such that:

$$|y_n^{k+1}(t) - y_n^k(t)| \leq L_K a_n^k(t).$$

Now $x_n^{k\prime} \in P_M F(x_n^k, y_n^k) \subset P_M F(x_n^k, y_n^{k+1}) + L L_K a_n^k \overline{B}$, hence by Filippov's theorem there exists x_n^{k+1} such that:

$$\begin{array}{l} x_n^{k+1\prime} \in P_M F(x_n^{k+1}, y_n^{k+1}), \\ x_n^{k+1}(0) = x, \\ |x_n^{k+1}(t) - x_n^k(t)| \leq L L_K e^{LT} \int_0^t a_n^k(s)\ ds \text{ for } t \in [0,T]. \end{array}$$

Calculating the integral we find that:

$$|x_n^{k+1}(t) - x_n^k(t)| \leq a_n^{k+1}(t),$$

which ends the recurrence.

Now we see that the sequences x_n^k, y_n^k, converge uniformly on $[0,T]$ towards, respectively, two functions $\overline{x}_n$, $\overline{y}_n$, with the estimation:

$$\begin{array}{ll} \sup_{t\in[0,T]} & |x_n(t) - \overline{x}_n(t)| \leq \rho\beta_n \quad \leq \rho L e^{LT} \displaystyle\int_0^T d_{K\left(x_n(s)\right)}\big(y_n(s)\big)\ ds, \\ \sup_{t\in[0,T]} & |y_n^1(t) - \overline{y}_n(t)| \leq \rho\beta_n, \end{array}$$

for a constant ρ depending on T. From Lemma 3.3 we know that:

$$\int_0^T d_{K(x_n(s))}\big(y_n(s)\big)\ ds \rightarrow_{n\rightarrow+\infty} 0,$$

hence $\beta_n \rightarrow_{n\rightarrow+\infty} 0$, from which we deduce that:

$$\sup_{t\in[0,T]} |x_n(t) - \overline{x}_n(t)| \rightarrow_{n\rightarrow+\infty} 0.$$

On the other hand we have:

$$\forall t \in [t_1, T],\ |y_n(t) - \overline{y}_n(t)| \leq d_{K\left(x_n(t)\right)}\big(y_n(t)\big) + \rho\beta_n,$$

and from Lemma 3.3 we know that:

$$\sup_{t\geq t_1} d_{K(x_n(t))}\big(y_n(t)\big) \rightarrow_{n\rightarrow+\infty} 0,$$

hence we deduce that:

$$\sup_{t\in[t_1,T]} |y_n(t) - \overline{y}_n(t)| \rightarrow_{n\rightarrow+\infty} 0.$$

The last assertion of the lemma is proved. The second assertion: $\overline{y}_n \in K(\overline{x}_n)$, is immediate by passing to the limit in (26). The first one: $\overline{x}_n' \in P_M F(\overline{x}_n, \overline{y}_n)$, is a consequence of (25), using the same convergence theorem as in Lemma 3.6. ∎

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