HARMONIC MORPHISMS BETWEEN WEYL SPACES AND TWISTORIAL MAPS

E. LOUBEAU, R. PANTILIE [1]

Abstract

We show that Weyl spaces provide a natural context for harmonic morphisms.

Introduction

Harmonic morphisms between Riemannian manifolds are smooth maps which pull back (local) harmonic functions to harmonic functions. By the basic characterisation theorem of B. Fuglede and T. Ishihara, harmonic morphisms are harmonic maps which are horizontally weakly conformal [7], [13].

The simplest nontrivial examples of harmonic morphisms are given by harmonic functions from a two-dimensional oriented conformal manifold: any such harmonic morphism is the sum of a $(+)$holomorphic function and a $(-)$holomorphic function (see [3]). Similar descriptions, in higher dimensions, can be obtained if instead of $(\pm)$holomorphic functions we use the more general notion of *twistorial map* [29]. A *twistorial structure* on a complex manifold M is given by a foliation $\mathcal{F}$ on a complex manifold P such that $\mathcal{F} \cap \ker \mathrm{d}\pi = \{0\}$ where $\pi : P \to M$ is a proper complex analytic submersion. It follows that, locally, we can find sections of π whose images are foliated by leaves of $\mathcal{F}$; by projecting back through π we endow M with a sheaf of complex analytic submersions. A *twistorial function* on M is a function which, locally, is the composition of such a submersion followed by a complex analytic function. Therefore any twistorial structure on M determines a sheaf of twistorial functions $\mathcal{F}_M$ on M; such sheaves can be obtained by complexifying the following examples of complex valued functions:

(i) holomorphic functions on a Hermitian manifold,

1991 *Mathematics Subject Classification.* Primary 58E20, Secondary 53C43.

Key words and phrases. harmonic morphism, Weyl space, twistorial map.

[1] Gratefully acknowledges that this work was done under a Grant from the Conseil Général du Finistère.

(ii) functions on an anti-self-dual 4-manifold which are holomorphic with respect to a (local) positive Hermitian structure on it,

(iii) functions on an Einstein–Weyl 3-manifold which are horizontally weakly conformal and whose regular fibres are geodesics.

In this paper, we work with twistorial maps which pull back twistorial functions to twistorial functions.

One of the main steps in the process of classifying harmonic morphisms is to prove that these are twistorial. For example, any harmonic morphism from an Einstein manifold of dimension four with fibres of dimension one or two is twistorial ([26] , [31]).

This paper attempts to give an answer to the following question of John C. Wood: *Can (the submersive) twistorial maps be seen as harmonic morphisms*? If we restrict ourselves to twistorial maps which pull back twistorial functions to twistorial functions then the answer, in the affirmative, to this question follows if we work with sheaves of twistorial functions $\mathcal{F}_M$ for which there exists a sheaf of 'harmonic' functions $\mathcal{L}$ such that $\mathcal{F}_M \cap \mathcal{L}$ is a 'sufficiently large' subsheaf of $\mathcal{L}$ (in particular, if the sheaf of vector spaces generated by $\mathcal{F}_M \cap \mathcal{L}$ is equal to $\mathcal{L}$, like in the case of two-dimensional conformal manifolds). We argue that for each of the examples (i) , (ii) , (iii) , above, a good candidate for $\mathcal{L}$ can be obtained by endowing the given conformal structure with a suitable Weyl connection (the obvious one, for (iii)).

The definition of harmonic functions on a Weyl space is given in Section 1 ; there we also show that the basic theorem of B. Fuglede and T. Ishihara generalizes to harmonic morphisms between Weyl spaces. In Section 2 , we do the same for the fundamental equation for harmonic morphisms (see [3]). In Section 3 , we recall the definition and the basic properties of the Weyl connection of an almost Hermitian manifold [30] ; we show that, for Hermitian manifolds, this is characterised by the property that all the ($\pm$)holomorphic functions are harmonic with respect to it. We also, prove that any holomorphic horizontally weakly conformal map between almost Hermitian manifolds endowed with their Weyl connections is harmonic and, hence, a harmonic morphism (cf. [20] , [11]). In Section 4 , we discuss harmonic morphisms from Weyl spaces of dimension three and four. We show that a map from a three-dimensional Weyl space to a two-dimensional conformal manifold is a harmonic morphism if and only if it is twistorial. The main results, of Section 4 , are the following:

• Any harmonic morphism from an Einstein–Weyl space of dimension four to a conformal manifold of dimension two is twistorial (Theorem 4.10 ; cf. [31]).

• Any harmonic morphism between Einstein–Weyl spaces of dimension four and three is twistorial (Theorem 4.11 ; cf. [26]).

Finally, in Remark 4.12 we explain how further consequences can be obtained, for a harmonic morphism between Einstein–Weyl spaces of dimension four and three, by using known facts from four-dimensional Weyl geometry.

We are grateful to Paul Baird for his interest in this work.

1. Harmonic morphisms between Weyl spaces

In this section we shall work in the smooth and (real or complex) analytic categories. A conformal manifold (M^m, c) is a manifold endowed with a reduction of its frame bundle to $CO(m, \mathbb{K})$, ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). We shall denote by L^2 the line bundle associated to the bundle of conformal frames of (M^m, c) via the morphism of Lie groups $\rho_m : CO(m, \mathbb{K}) \to \mathbb{K} \setminus \{0\}$ characterised by $a^T a = \rho_m(a) I_m$, ($a \in CO(m, \mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$). The notation is motivated by the fact that in the smooth and real analytic categories L^2 admits a square root, denoted by L , which does not depend of c (see [5]). In the complex analytic category such a square root can be found locally. Furthermore, in the smooth and real analytic categories L^2 depends only of M^m and is oriented; then positive local sections of L^2 correspond to local representatives of c. In the complex analytic category, nowhere zero local sections of L^2 correspond to local representatives of c. Note that, if b is a section of $\otimes^2 T^*M$ then its traces with respect to local representatives of c define a section of L^2 which will be denoted $\mathrm{trace}_c\, b$. More generally, if b is a section of $E \otimes (\otimes^2 T^*M)$ for some vector bundle E over M then we can define $\mathrm{trace}_c\, b$ which is a section of $E \otimes L^2$; if $E = TM$ then $(\mathrm{trace}_g\, b)^\flat$, where g is any local representative of c, defines a 1-form on M which will be denoted $(\mathrm{trace}_c\, b)^\flat$ (see [8] , [5]).

The following definition is a simple generalization of the well-known notion of harmonic map (see [3]).

Definition 1.1. Let (M,c) be a conformal manifold, N a manifold and D^M, D^N linear connections on M, N, respectively.

A map $\varphi:(M,c,D^M)\to(N,D^N)$ is called *harmonic (with respect to c, D^M, D^N)* if

$$\operatorname{trace}_c(D\mathrm{d}\varphi)=0 \tag{1.1}$$

where D is the connection on $\varphi^*(TN)\otimes T^*M$ induced by D^M, D^N and φ.

Obviously, there is no loss of generality if we assume D^M and D^N to be torsion free.

A harmonic map $\varphi:(M,g)\to(N,h)$ between Riemannian manifolds is harmonic in the sense of Definition 1.1 if M and N are endowed with the Levi-Civita connections of g and h, respectively, and M is considered with the conformal structure determined by g.

We shall always consider $\mathbb{K}(=\mathbb{R},\mathbb{C})$ to be endowed with its canonical conformal structure and trivial connection (here $\mathbb{C}$ is considered to be a one-dimensional complex manifold). Clearly, a curve on (M,D) is harmonic if and only if it is a geodesic of D.

Let (M,c) be a conformal manifold. A torsion free conformal connection on (M,c) is called a *Weyl connection*; if D is a Weyl connection on (M,c) then (M,c,D) is called a *Weyl space* (see [8]). A function (locally) defined on a Weyl space (M,c,D) will be called *harmonic* if it is harmonic with respect to c, D. If $\dim M=2$ then a function f on the Weyl space (M,c,D) is harmonic if and only if it is harmonic with respect to any local representative of c.

Proposition 1.2. *Let (M,c_M) be a conformal manifold, of dimension $m\neq 2$, endowed with a linear connection D.*

Then there exists a unique Weyl connection D_1 on (M,c_M) such that

$$\operatorname{trace}_{c_M}(D\mathrm{d}f)=\operatorname{trace}_{c_M}(D_1\mathrm{d}f) \tag{1.2}$$

for any function f (locally) defined on M.

Proof. For each local representative g of c_M we define a (local) 1-form α^g by

$$\alpha^g(X) = \frac{1}{m-2}\, g(\mathrm{trace}_g(\nabla^g - D), X) \tag{1.3}$$

for all $X \in TM$, where ∇^g is the Levi-Civita connection of g. It is easy to prove that $\alpha^{g\lambda^{-2}} = \alpha^g + \lambda^{-1}\mathrm{d}\lambda$. Hence, the family of 1-forms $\{\alpha^g\}$ defines a connection on L. But any connection on L corresponds to a Weyl connection D_1 on (M, c_M) (see [8], the 1-form α^g is *the Lee form of* D_1 *with respect to* g). Now, (1.2) is equivalent to (1.3) and the proof follows. □

The following definition (cf. [7], [13], [3]) will be central in this paper.

Definition 1.3. Let (M, c_M, D^M) and (N, c_N, D^N) be Weyl manifolds.

A map $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ is called a *harmonic morphism* if for any harmonic function f defined on some open set U of N, such that $\varphi^{-1}(U) \neq \emptyset$, the function $f \circ \varphi|_{\varphi^{-1}(U)}$ is harmonic.

Remark 1.4. Proposition 1.2 shows that, if $\dim M, \dim N \neq 2$ then Definition 1.3 does not become more general by using linear connections instead of Weyl connections.

Any harmonic morphism between Riemannian manifolds $\varphi : (M, g) \to (N, h)$ is also a harmonic morphism between Weyl spaces $\varphi : (M, [g], \nabla^g) \to (N, [h], \nabla^h)$, where $[g]$, $[h]$ are the conformal structures determined by g, h and ∇^g, ∇^h are the Levi-Civita connections of g, h, respectively. However, not all harmonic morphisms between Weyl spaces arise in this way (see Sections 3 and 4).

Next we shall prove the Fuglede-Ishihara theorem ([7], [13], see [3]) for harmonic morphisms between Weyl spaces. For this we apply the standard strategy (see [3]). Firstly, we need an existence result for harmonic functions from Weyl spaces:

Lemma 1.5 (cf. [3]). *Let (M, c, D) be a Weyl space and let $x \in M$.*

Then for any $v \in T^*_xM$ and any trace free symmetric bilinear form b on (T_xM, c_x) there exists a harmonic function f defined on some open neighbourhood of x such that $\mathrm{d}f_x = v$ and $(D\mathrm{d}f)_x = b$.

Proof. This is essentially the same as for harmonic functions on Riemannian manifolds (see [3] and the references therein).

We shall give a straightforward proof assuming (M,c,D) (real or complex) analytic (cf. [21, Lemma 2], where we assume that the metric is analytic). Let U be the domain of a normal coordinate system $x^1,\dots,x^m$ for D, centred at x, where $m=\dim M$. We may assume $g(\mathrm{d}x^m,\mathrm{d}x^m)=1$, at x, for some local representative g of c over U. Hence, by passing to a smaller open neighbourhood of x, if necessary, we may assume that the hypersurface $S=\{x^m=0\}$ is nowhere degenerate; equivalently, S is noncharacteristic for the second order differential operator $f\mapsto \operatorname{trace}_g(D\mathrm{d}f)$.

Let $p=b_{ij}x^ix^j+v_ix^i$. Then, by further restricting U, if necessary, and by applying the Cauchy-Kovalevskaya theorem, we can find a harmonic function f, with respect to c,D, defined on U such that f and p are equal up to the first derivatives along S; in particular, $\mathrm{d}f_x=v$. Hence, possibly excepting $\frac{\partial^2 f}{(\partial x^m)^2}(x)$, all the second order partial derivatives of f, at x, are equal to the corresponding derivatives of p, at x. As f is harmonic, b is trace free, with respect to g, and x is the centre of the normal system of coordinates $x^1,\dots,x^m$, for D, the derivatives $\frac{\partial^2 f}{(\partial x^m)^2}(x)$ and $\frac{\partial^2 p}{(\partial x^m)^2}(x)$ are determined by the other second order partial derivatives, at x, of f and p, respectively, and hence must be equal. Thus $(D\mathrm{d}f)_x=b$. □

Remark 1.6. Let f be a harmonic function (locally defined) on a Weyl space (M,c,D) and let $x\in M$ such that $\mathrm{d}f_x\neq 0$. Then there exists a local representative g of c defined on some neighbourhood U of x such that f is harmonic with respect to g (this follows, for example, from (1.3) applied to D). However, the sheaf of harmonic functions on U, with respect to c, D, is equal to the sheaf of harmonic functions of g if and only if D is the Levi-Civita connection of g.

Now we can prove the Fuglede-Ishihara theorem for harmonic morphisms between Weyl spaces.

Theorem 1.7 (cf. [7], [13]). *Let (M,c_M,D^M) and (N,c_N,D^N) be Weyl manifolds.*

A map $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ is a harmonic morphism if and only if it is harmonic, with respect to c_M, D^M, D^N, and horizontally weakly conformal (that is, at each point $x\in M$ either $\mathrm{d}\varphi_x=0$ or its adjoint with respect to any representatives of c^M and c^N is conformal; in particular $\dim M\geq\dim N$ if φ is nonconstant).

Proof. Let g be a representative of c_M over some open set U of M and let h be a representative of c_N over some set V of N such that $\varphi(U) \subseteq V$. For any function (locally) defined on U, a straightforward calculation gives (cf. [3])

$$\operatorname{trace}_g\big(D\mathrm{d}(f\circ\varphi)\big) = \mathrm{d}f\big(\operatorname{trace}_g(D\mathrm{d}\varphi)\big) + h\big(D\mathrm{d}f, ((\mathrm{d}\varphi)^T)^*(g)\big) \tag{1.4}$$

where $(\mathrm{d}\varphi)^T$ denotes the adjoint of $\mathrm{d}\varphi$ with respect to g and h.

By applying Lemma 1.5 with $b=0$ and for all $v \in T_x^*N$, $(x \in N)$, from equation (1.4) we obtain that φ is harmonic with respect to c_M, D^M, D^N. Then, by applying again Lemma 1.5, from equation (1.4) we obtain that for all trace free symmetric $b \in \otimes^2 T^*N$ we have $h\big(b, ((\mathrm{d}\varphi)^T)^*(g)\big) = 0$. It follows that there exists a function Λ on U such that $((\mathrm{d}\varphi)^T)^*(g) = \Lambda\, h$; that is, $(\mathrm{d}\varphi)^T$ is conformal with conformal factor Λ. □

Remark 1.8 (see [3] , [5]). Let $\varphi : (M, c_M) \to (N, c_N)$ be a horizontally weakly conformal map between conformal manifolds. The conformal factors of $(\mathrm{d}\varphi)^T$ with respect to local representatives of c_M and c_N define a section Λ of $\mathrm{Hom}(\varphi^*(L_N^2), L_M^2)$ which is zero over the critical points of φ, where L_M^2, L_N^2 are the line bundles associated to (M, c_M), (N, c_N), respectively; we shall call this section the *square dilation* of φ.

In the smooth and real analytic categories, if φ is submersive, $\Lambda = \lambda^2$ for a unique positive section λ of $\mathrm{Hom}(\varphi^*(L_N), L_M)$. We shall call λ the *dilation* of φ. In the complex analytic category, once we have locally chosen L_M and L_N, the dilation of the horizontally conformal submersion φ is well-defined locally, up to sign.

If φ is not submersive then its dilation can be defined as a continuous (local) section of $\mathrm{Hom}(\varphi^*(L_N), L_M)$ which is zero over the set of critical points of φ.

See [3] , [23] for more information on horizontally weakly conformal maps between Riemannian manifolds.

2. The fundamental equation

In this section we shall work in the smooth and (real or complex) analytic categories. Let (M^m, c_M) be a conformal manifold and let L be the corresponding line bundle on M. Let $\mathcal{V} \subseteq TM$ be a nondegenerate distribution, and let $\mathcal{H} = \mathcal{V}^\perp$ be its orthogonal complement. Then c_M induces conformal structures

$c_M|_{\mathcal{V}}$ and $c_M|_{\mathcal{H}}$ on $\mathcal{V}$ and $\mathcal{H}$, respectively. Let $L_{\mathcal{V}}$ and $L_{\mathcal{H}}$ be the line bundles on M determined by the conformal structures $c_M|_{\mathcal{V}}$ and $c_M|_{\mathcal{H}}$, respectively. As any local representative of c_M induces local representatives of the conformal structures induced on $\mathcal{V}$ and $\mathcal{H}$, we have isomorphisms between L^2, $L^2_{\mathcal{V}}$ and $L^2_{\mathcal{H}}$ (seen as bundles with group $\big((0,\infty),\cdot\big)$, in the smooth and real analytic categories); we shall always identify $L^2 = L^2_{\mathcal{V}} = L^2_{\mathcal{H}}$, in this way. Conversely, conformal structures on the complementary distributions $\mathcal{V}$ and $\mathcal{H}$ together with an isomorphism between $L^2_{\mathcal{V}}$ and $L^2_{\mathcal{H}}$ determine a conformal structure on M such that $\mathcal{H} = \mathcal{V}^{\perp}$ [5]. In other words, nondegenerate distributions $\mathcal{V}$, of dimension $m-n$, on (M^m, c_M) correspond to reductions of c_M to the subgroup $G = \big\{ (a,b) \in CO(m-n) \times CO(n) \,|\, \rho_{m-n}(a) = \rho_n(b) \big\}$ of $CO(m)$. Then, as the morphisms of Lie groups $p_1 : G \to CO(m-n)$, $(a,b) \mapsto a$, and $p_2 : G \to CO(n)$, $(a,b) \mapsto b$, satisfy $\rho_m|_G = \rho_{m-n} \circ p_1 = \rho_n \circ p_2$, we obtain that $c_M|_{\mathcal{V}} = p_1(c_M)$ and $c_M|_{\mathcal{H}} = p_2(c_M)$ are such that $L^2_{\mathcal{V}}$ and $L^2_{\mathcal{H}}$ are isomorphic to L^2. Conversely, if $c_{\mathcal{V}}$ and $c_{\mathcal{H}}$ are (the bundles of conformal frames of) conformal structures on the complementary distributions $\mathcal{V}$ and $\mathcal{H}$, respectively, then $c_{\mathcal{V}} + c_{\mathcal{H}}$ is a reduction of the bundle of linear frames on M^m to $CO(m-n) \times CO(n)$ and, it is easy to see that, isomorphisms between $L^2_{\mathcal{V}}$ and $L^2_{\mathcal{H}}$ correspond to reductions of $L^2_{\mathcal{V}} \oplus L^2_{\mathcal{H}}$ to $\iota : H \hookrightarrow H \times H$, $a \mapsto (a,a)$, where $H = \big((0,\infty),\cdot\big)$ in the smooth and real analytic categories, and $H = \big(\mathbb{C} \setminus \{0\},\cdot\big)$ in the complex analytic category. As $G = (\rho_{m-n} \times \rho_n)^{-1}(\iota(H))$, it follows that reductions of $c_{\mathcal{V}} + c_{\mathcal{H}}$ to G correspond to isomorphisms between $L^2_{\mathcal{V}}$ and $L^2_{\mathcal{H}}$; any such reduction determines a conformal structure c_M on M^m such that $\mathcal{H} = \mathcal{V}^{\perp}$ and $c_M|_{\mathcal{V}} = c_{\mathcal{V}}$, $c_M|_{\mathcal{H}} = c_{\mathcal{H}}$.

Example 2.1 ([4]). Let M be a manifold endowed with two complementary distributions $\mathcal{V}$ and $\mathcal{H}$. The *Bott partial connection* D^{Bott} on $\mathcal{V}$, over $\mathcal{H}$, is defined by $D^{\mathrm{Bott}}_X U = \mathcal{V}[X,U]$ for local sections X of $\mathcal{H}$ and U of $\mathcal{V}$.

Suppose that M is endowed with a conformal structure c_M with respect to which $\mathcal{V}$ is nondegenerate and $\mathcal{H} = \mathcal{V}^{\perp}$. As $(L^2)^{m-n} = (\Lambda^{m-n}\mathcal{V})^2$, where n is the dimension of the distribution $\mathcal{H}$, D^{Bott} induces a partial connection on L which will also be denoted D^{Bott}; the local connection form of this connection with respect to a local section of L, corresponding to a local representative g of c_M, is $\frac{1}{m-n}\,\mathrm{trace}_g(B^{\mathcal{V}})^{\flat}$ (cf. Example 2.5, below).

Let $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ be a horizontally conformal submersion with nowhere degenerate fibres between Weyl spaces. We shall denote, as usual (see, for example, [3]), $\mathcal{V} = \ker \mathrm{d}\varphi$, $\mathcal{H} = \mathcal{V}^\perp$. Then D^M and D^N induce Weyl partial connections, with respect to $\mathcal{V}$, on $(\mathcal{H}, c_M|_{\mathcal{H}})$, over $\mathcal{H}$, which will be denoted $\mathcal{H}D^M$ and D^N, respectively. (Recall (see [29]) that a Weyl partial connection D on $(\mathcal{H}, c)$, over $\mathcal{H}$, is a conformal partial connection D on $(\mathcal{H}, c)$ whose torsion tensor field T, with respect to $\mathcal{V}$, defined by $T(X,Y) = D_XY - D_YX - \mathcal{V}[X,Y]$ for local sections X and Y of $\mathcal{H}$, is zero.)

If D is a (partial) connection on L and $k \in \mathbb{Z}$ then we shall denote by D^k the (partial) connection induced on $L^k \,(= \otimes^k L)$.

Proposition 2.2. *Let $\varphi : (M^m, c_M, D^M) \to (N^n, c_N, D^N)$ be a horizontally conformal submersion with nowhere degenerate fibres between Weyl spaces. Then*

$$\operatorname{trace}_{c_M}(D\mathrm{d}\varphi)^\flat = (\mathcal{H}D^M)^{m-2} \otimes (D^N)^{-(n-2)} - (D^{\mathrm{Bott}})^{m-n} \,. \tag{2.1}$$

Proof. Let $B^{\mathcal{V},D^M}$ be the second fundamental form of $\mathcal{V}$, with respect to D^M, defined by

$$B^{\mathcal{V},D^M}(U,V) = \frac{1}{2}\,\mathcal{V}(D_UV + D_VU)$$

for local sections U and V of $\mathcal{V}$ (see [5] , cf. [3]). A straightforward calculation gives

$$\operatorname{trace}_{c_M}(D\mathrm{d}\varphi)^\flat = \operatorname{trace}_{c_M}(D^N - \mathcal{H}D^M) - \operatorname{trace}_{c_M}(B^{\mathcal{V},D^M})^\flat \,. \tag{2.2}$$

Now let g be a local representative of c_M, corresponding to some local section s of L, and let α_M and α_N be the Lee forms of D^M and D^N, respectively, with respect to g. Recall (see [8]) that α_M (α_N) is the local connection form of D^M (D^N) with respect to s. Also, it is easy to prove that

$$\operatorname{trace}_g(B^{\mathcal{V},D^M})^\flat = \operatorname{trace}_g(B^{\mathcal{V}})^\flat - (m-n)\alpha_M|_{\mathcal{H}} \tag{2.3}$$

where $B^{\mathcal{V}}$ is the second fundamental form of $\mathcal{V}$ with respect to (the Levi-Civita connection of) g.

It follows that (2.2) is equivalent to

$$\operatorname{trace}_{c_M}(D\mathrm{d}\varphi)^\flat = (m-2)\alpha_M|_{\mathcal{H}} - (n-2)\alpha_N - \operatorname{trace}_g(B^{\mathcal{V}})^\flat \tag{2.4}$$

The proof follows from Example 2.1 . □

Remark 2.3. 1) When D^M and D^N are the Levi-Civita connections of (local) representatives of c_M and c_N, respectively, then (2.1) reduces to the *fundamental equation* for horizontally conformal submersions (see [3]).

2) From the fundamental equation (2.1) it follows that if $\varphi : (M, c_M) \to (N, c_N, D^N)$ is a horizontally conformal submersion, with nowhere degenerate fibres, from a conformal manifold to a Weyl space then there exists a Weyl connection D^M on (M, c_M) such that $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ is a harmonic morphism.

We shall say that $\mathcal{V}$ is *minimal, with respect to* D^M, if $\mathrm{trace}_{c_M}(B^{\mathcal{V},D^M}) = 0$; then $\mathcal{V}$ is minimal, with respect to D^M, if and only if D^M and D^{Bott} induce the same connection on L.

Similar to the case of harmonic morphisms between Riemannian manifolds, from the *fundamental equation* (2.1), we obtain the following.

Theorem 2.4 (cf. [1]). *Let $\varphi : (M^m, c_M, D^M) \to (N^n, c_N, D^N)$ be a horizontally conformal submersion with nowhere degenerate fibres between Weyl spaces.*

(a) *If* $\dim N = 2$ *then φ is a harmonic morphism if and only its fibres are minimal, with respect to D^M.*

(b) *If* $\dim N \neq 2$ *then any two of the following assertions imply the third:*

(i) *φ is a harmonic morphism.*

(ii) *The fibres of φ are minimal, with respect to D^M.*

(iii) $\mathcal{H}D^M = D^N$. □

We end this section with an example of a Weyl connection which will be useful later on.

Example 2.5 ([5]). Let (M^m, c) be a conformal manifold endowed with a non-degenerate distribution $\mathcal{V}$, of codimension n, and let $\mathcal{H} = \mathcal{V}^{\perp}$.

For each local representative g of c_M define a (local) 1-form α^g by

$$\alpha^g = \frac{1}{m-n}\,\mathrm{trace}_g(B^{\mathcal{V}})^{\flat} + \frac{1}{n}\,\mathrm{trace}_g(B^{\mathcal{H}})^{\flat}\,. \tag{2.5}$$

Then $\alpha^{g\lambda^{-2}} = \alpha^g + \lambda^{-1}\mathrm{d}\lambda$. Hence, the family of 1-forms $\{\alpha^g\}$ defines a Weyl connection D on (M^m, c). The Weyl connection D is called the *(minimal) Weyl connection of* $(M^m, c, \mathcal{V})$.

As (2.3) holds without the assumption that $\mathcal{V}$ is conformal, $\mathcal{V}$ and $\mathcal{H}$ are minimal with respect to D; moreover, D is the unique Weyl connection on (M^m, c) with this property. It follows that if $\mathcal{V}$ is one-dimensional and conformal then the connection induced by D on L is flat if and only if $\mathcal{V}$ is locally generated by conformal vector fields.

3. Harmonic maps and morphisms between almost Hermitian manifolds

In this section we shall work in the smooth and (real or complex) analytic categories. An *almost Hermitian (conformal) manifold* is a triple (M, c, J) where (M, c) is a conformal manifold and J is a compatible almost complex structure; that is, if we consider c as an L^2-valued Riemannian metric on M [8] then we have $c(JX, JY) = c(X, Y)$, $(X, Y \in TM)$. Therefore, $\dim M$ is even and the *Kähler form of* (M, c, J), defined by, $\omega(X, Y) = c(JX, Y)$, $(X, Y \in TM)$, is an L^2-valued almost symplectic structure on M. A *Hermitian (conformal) manifold* is an almost Hermitian manifold (M, c, J) such that J is integrable.

To any almost Hermitian manifold, of dimension at least four, can be associated, in a natural way, a Weyl connection, as follows.

Proposition 3.1 ([30]). *Let (M, c, J) be an almost Hermitian manifold, of dimension $m \geq 4$, and let $\omega \in \Gamma(L^2 \otimes \Lambda^2 T^* M)$ be its Kähler form.*

There exists a unique Weyl connection D on (M, c) such that $\mathrm{trace}_c(DJ) = 0$, *the Lee form of D with respect to a local representative g of c, is equal to $-\frac{1}{m-2}$ times the Lee form of J with respect to g.*

Proof. Let $m = 2n$, $(n \geq 2)$. From the fact that ω is an L^2-valued almost symplectic structure on M, it follows (see [5]) that there exists a unique connection D on L^2 such that

$$\mathrm{d}^D \omega \wedge \omega^{n-2} = 0 \; . \tag{3.1}$$

We shall denote by the same letter D the induced connection on L and the corresponding Weyl connection on (M, c). Let s be a local section of L and let ω^s be the Kähler form of J with respect to the local representative g^s of c corresponding to s; that is, $\omega^s(X, Y) = g^s(JX, Y)$, $(X, Y \in TM)$. It is easy to prove that (3.1) is equivalent to the fact that, for any local section s of L, the local connection form of D, with respect to s, is equal to $-\frac{1}{m-2}$ times the Lee form of J, with respect to g^s.

Furthermore, (3.1) is also equivalent to $\sum_{i=1}^{n}(\mathrm{d}^D\omega)(X_i, JX_i, \dot{)} = 0$ for any conformal frame $\{X_1, JX_1, \dots, X_n, JX_n\}$. Therefore to end the proof it is sufficient to show that for any Weyl connection D on (M, c) we have

$$\sum_{i=1}^{n}(\mathrm{d}^D\omega)(X_i, JX_i, JY) = -c(\mathrm{trace}_g(DJ), Y)$$

for any $Y \in TM$ and where g is the metric determined by $\{X_1, JX_1, \cdots, X_n, JX_n\}$. Indeed, as $Dc = 0$ we have $(D\omega)(X, Y) = c((DJ)(X), Y)$ and $(D\omega)(X, JX) = 0$, $(X, Y \in TM)$. Therefore

$$\begin{aligned}\sum_{i=1}^{n}(\mathrm{d}^D\omega)(X_i, JX_i, JY) &= (D_{X_i}\omega)(JX_i, JY) + (D_{JX_i}\omega)(JY, X_i) + (D_{JY}\omega)(X_i, JX_i)\\ &= c((D_{X_i}J)(JX_i), JY) + c((D_{JX_i}J)(JY), X_i)\\ &= -c(J(D_{X_i}J)(X_i), JY) - c(J(D_{JX_i}J)(Y), X_i)\\ &= -c((D_{X_i}J)(X_i), Y) - c(Y, (D_{JX_i}J)(JX_i))\\ &= -c(\mathrm{trace}_g(DJ), Y)\,.\end{aligned}$$

□

Definition 3.2 ([30]). Let (M, c, J) be an almost Hermitian manifold, of dimension $\dim M \geq 4$.

The *Weyl connection of* (M, c, J) is the Weyl connection D on (M, c) such that $\mathrm{trace}_c(DJ) = 0$.

Remark 3.3. 1) [30] Let (M, c, J) be an almost Hermitian manifold, of dimension $\dim M \geq 4$, and let D be a Weyl connection on (M, c).

Let ∇ be the Levi-Civita connection of a local representative g of c. Then $D_{JX}J - J\,D_XJ = \nabla_{JX}J - J\,\nabla_XJ$, $(X \in TM)$. Hence J is integrable if and only if $D_{JX}J = J\,D_XJ$, $(X \in TM)$.

On the other hand, the condition $D_{JX}J = -J\,D_XJ$, $(X \in TM)$, is equivalent to $(\mathrm{d}^D\omega)^{(1,2)\oplus(2,1)} = 0$ and is a sufficient condition for D to be the Weyl connection of (M, c, J). Hence, if $\dim M = 4$ then $D_{JX}J = -J\,D_XJ$, $(X \in TM)$, if and only if D is the Weyl connection of (M, c, J).

Thus, if $\dim M = 4$ then $DJ = 0$ if and only if J is integrable and D is the Weyl connection of (M, c, J). If $\dim M \geq 6$ then it follows that $DJ = 0$ if and only if, locally, there exist representatives g of c with respect to which (M, g, J) is Kähler.

2) Let (M, c, J) be an almost Hermitian manifold and let f be a $(\pm)$holomorphic function locally defined on (M, J). (If (M, c, J) is complex analytic then by a $(\pm)$holomorphic function we mean a function which is constant along curves tangent to the $(\mp\mathrm{i})$eigendistributions of J.) If $\dim M = 2$ then f is harmonic with respect to any local representative of c (see [3]). If $\dim M \geq 4$ and D is the Weyl connection of (M, c, J) then f is a harmonic function of (M, c, D).

Furthermore, if (M, c, J) is a Hermitian manifold, $\dim M \geq 4$, then for any Weyl connection D on (M, c) the following assertions are equivalent:

(i) D is the Weyl connection of (M, c, J).

(ii) Any $(\pm)$holomorphic function of (M, J) is a harmonic function of (M, c, D).

See Proposition 3.6 for a reformulation of this equivalence, in the complex analytic category.

Next, we prove the following useful lemma.

Lemma 3.4 (cf. [20] , [27]). *Let D^M, D^N be torsion free connections on the almost complex manifolds (M, J^M), (N, J^N), respectively. Suppose that M is endowed with a conformal structure c and let $\varphi : (M, J^M) \to (N, J^N)$ be a holomorphic map. Then*

$$\operatorname{trace}_c \varphi^*(D^N J^N) - \mathrm{d}\varphi(\operatorname{trace}_c(D^M J^M)) + J^N(\operatorname{trace}_c(D\mathrm{d}\varphi)) = 0 \ . \tag{3.2}$$

Proof. It is easy to prove that, for $X, Y \in TM$, we have

$$D\mathrm{d}\varphi(X, J^M Y) = (D^N_{\mathrm{d}\varphi(X)} J^N)(\mathrm{d}\varphi(Y)) - \mathrm{d}\varphi((D^M_X J^M)(Y)) + J^N(D\mathrm{d}\varphi(X, Y)) \ .$$

The proof follows. □

From Lemma 3.4 we easily obtain the following proposition (cf. Remark 3.3(2)).

Proposition 3.5 (cf. [20] , [11]). *Let (M, c_M, J^M) and (N, c_N, J^N) be almost Hermitian manifolds. If $\dim M \geq 4$, $\dim N \geq 4$ let D^M , D^N be the Weyl connections of (M, c_M, J^M) , (N, c_N, J^N) , respectively; if $\dim M = 2$ or $\dim N = 2$ then D^M or D^N will denote any Weyl connection on (M, c_M) or (N, c_N) , respectively.*

Let $\varphi:(M,J^M)\to(N,J^N)$ *be a holomorphic map.*

(i) *If* $(\mathrm{d}^{D^N}\omega_N)^{(1,2)\oplus(2,1)}=0$ *, where* ω_N *is the Kähler form of* (N,c_N,J^N) *, then* $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ *is a harmonic map.*

(ii) *If the map* $\varphi:(M,c_M)\to(N,c_N)$ *is horizontally weakly conformal then* $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ *is a harmonic map and hence a harmonic morphism.* □

Note that the assumption of assertion (i) of Proposition 3.5 is automatically satisfied if $\dim N=2\,,4$. Also, in Proposition 3.5(ii) the horizontally weakly-conformal map φ may have degenerate fibres. In fact, the result of Proposition 3.5(ii) can be extended as follows.

Proposition 3.6. *Let* (M,c_M,J^M) *be a complex analytic almost Hermitian manifold. If* $\dim M\geq 4$ *let* D^M *be the Weyl connection of* (M,c_M,J^M) *; if* $\dim M=2$ *let* D^M *be any Weyl connection on* (M,c_M) *.*

Let $\varphi:(M,c_M)\to N$ *be a horizontally conformal submersion with degenerate fibres such that* $\ker\mathrm{d}\varphi$ *contains* $\mathcal{F}$ *or* $\widetilde{\mathcal{F}}$ *, where* $\mathcal{F}$*,* $\widetilde{\mathcal{F}}$ *are the eigendistributions of* J^M *.*

Then $\varphi:(M,c_M,D^M)\to(N,D^N)$ *is a harmonic map with respect to any connection* D^N*, on* N*, and* $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ *is a harmonic morphism with respect to any structure of Weyl space on* N*.*

Conversely, if $\dim M\geq 4$ *,* J^M *is integrable and* D *is a Weyl connection on* (M,c_M) *such that the foliations* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *are locally defined by harmonic maps, with respect to* c_M*,* D *, then* D *is the Weyl connection of* (M,c_M,J^M) *.*

Proof. Suppose that $\mathcal{F}\subseteq\ker\mathrm{d}\varphi$. Then for any function f, locally defined on N, the function $f\circ\varphi$ is a holomorphic function of (M,J^M) . By Remark 3.3(2) , $f\circ\varphi$ is a harmonic function of (M,c_M,D^M) and hence $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ is a harmonic morphism with respect to any structure of Weyl space on N.

The second statement follows from the implication (ii)$\Rightarrow$(i) of Remark 3.3(2) . □

Let (N^2,c_N) be a two-dimensional oriented conformal manifold. Then there exists a complex structure J^N, uniquely determined up to sign, with respect to which (N^2,c_N,J^N) is a Hermitian manifold.

Let (M^4,c_M) be a four-dimensional complex analytic oriented conformal manifold. An *(anti-)self-dual plane at* $x\in M$ is a two-dimensional vector space

$p \subseteq T_xM$ such that for some (and hence any) basis $\{X,Y\}$ of p the 2-form $X \wedge Y$ is (anti-)self-dual; if (M^4, c_M) is an oriented smooth or real analytic manifold then an *(anti-)self-dual plane at* $x \in M$ is an (anti-)self-dual subspace of $(T_x^{\mathbb{C}}M, (c_M)_x^{\mathbb{C}})$ (see [22]).

Let (M^4, c_M) be a four-dimensional oriented conformal manifold endowed with a two-dimensional nondegenerate distribution $\mathcal{V}$. Then there exists an almost complex structure J^M, uniquely determined up to sign, with respect to which (M^4, c_M, J^M) is a positive almost Hermitian manifold such that $J^M\mathcal{V} = \mathcal{V}$. (We say that (M^4, c_M, J^M) is *positive* if some (and hence, any) conformal frame of the form $(X_1, J^MX_1, X_2, J^MX_2)$ is positive; equivalently, at some (and hence, any) point, the eigenspaces of J^M are self-dual. Note that, in the smooth and real analytic categories, this just means that J^M is a positive almost complex structure on M^4.) It follows that the Weyl connection of (M^4, c_M, J^M) is equal to

$$D - \frac{1}{2}\big(J^M(*_{\mathcal{V}}I^{\mathcal{V}})\big)^{\flat} - \frac{1}{2}\big(J^M(*_{\mathcal{H}}I^{\mathcal{H}})\big)^{\flat}$$

where D is the Weyl connection of $(M^4, c_M, \mathcal{V})$, $I^{\mathcal{V}}$, $I^{\mathcal{H}}$ are the integrability tensors of $\mathcal{V}$, $\mathcal{H}$, respectively, and the Hodge star-operators $*_{\mathcal{V}}$, $*_{\mathcal{H}}$, of $\mathcal{V}$, $\mathcal{H}$, respectively, and the musical isomorphism $^{\flat} : TM \to T^*M$ are all considered with respect to the same, arbitrarily chosen, local representative of c_M [5] ; equivalently, the Lee form of J^M with respect to any local representative g of c_M is equal to

$$-\operatorname{trace}_g(B^{\mathcal{V}})^{\flat} - \operatorname{trace}_g(B^{\mathcal{H}})^{\flat} + \big(J^M(*_{\mathcal{V}}I^{\mathcal{V}})\big)^{\flat} + \big(J^M(*_{\mathcal{H}}I^{\mathcal{H}})\big)^{\flat} .$$

Let $\varphi : (M^4, c_M) \to (N^2, c_N)$ be a horizontally conformal submersion, with nowhere degenerate fibres, between oriented conformal manifolds. Then there exists a unique almost complex structure J^M on M^4 with respect to which the map $\varphi : (M^4, J^M) \to (N^2, J^N)$ is holomorphic and (M^4, c_M, J^M) is a positive almost Hermitian manifold. Let D^M be a Weyl connection on (M^4, c_M).

Proposition 3.7 (cf. [31]). *The following assertions are equivalent:*

(i) *The map* $\varphi : (M^4, c_M, D^M) \to (N^2, c_N)$ *is a harmonic morphism and* J^M *is integrable.*

(ii) *The almost complex structure* J^M *is parallel along the fibres of* φ, *with respect to* D^M*; that is,* $D^M_U J^M = 0$, $(U \in \ker \mathrm{d}\varphi)$.

Proof. Firstly, we shall write the proof in the complex analytic category. Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be the eigendistributions of J^M. Then J^M is integrable if and only if $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are integrable. Also, note that, assertion (ii) holds if and only if $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are parallel, with respect to D^M, along $\mathcal{V}\,(=\ker \mathrm{d}\varphi)$.

Let f and $\widetilde{f}$ be the components of φ with respect to null local coordinates on (N^2, c_N). From the fact that $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal it follows that $\varphi : (M^4, c_M, D^M) \to (N^2, c_N)$ is harmonic if and only if f and $\widetilde{f}$ are harmonic functions of (M^4, c_M, D^M). Also, we may suppose $\mathcal{F} \subseteq \ker \mathrm{d}f$, $\widetilde{\mathcal{F}} \subseteq \ker \mathrm{d}\widetilde{f}$.

There exists a local frame $\{U, \widetilde{U}, Y, \widetilde{Y}\}$ on M^4 such that $g = 2(U \odot \widetilde{U} + Y \odot \widetilde{Y})$ is a local representative of c_M, $U, \widetilde{U}$ are vertical, $Y, \widetilde{Y}$ are horizontal, and $\{U, Y\}$, $\{\widetilde{U}, \widetilde{Y}\}$ are local frames of $\mathcal{F}$, $\widetilde{\mathcal{F}}$, respectively.

As $\{U, \widetilde{U}, Y\}$ is a local frame of $\ker \mathrm{d}f$ we have $g([U, Y], Y) = 0$. Hence, $\mathcal{F}$ is integrable if and only if $g([U, Y], U) = 0$. As $g([U, Y], U) = g(D^M_U Y, U)$ we obtain that $\mathcal{F}$ is integrable if and only if $\mathcal{F}$ is parallel along U, with respect to D^M.

Also, $\mathrm{trace}_g(D^M \mathrm{d}f) = -2g(D^M_U \widetilde{U}, Y)\, \widetilde{Y}(f)$. Hence, f is a harmonic function of (M^4, c_M, D^M) if and only if $\mathcal{F}$ is parallel along $\widetilde{U}$, with respect to D^M.

Therefore, $\mathcal{F}$ is integrable and f is harmonic if and only if $\mathcal{F}$ is parallel, with respect to D^M, along $\mathcal{V}$. Similarly, $\widetilde{\mathcal{F}}$ is integrable and $\widetilde{f}$ is harmonic if and only if $\widetilde{\mathcal{F}}$ is parallel, with respect to D^M, along $\mathcal{V}$. Thus the proof is complete, in the complex analytic category.

In the smooth and real analytic categories essentially the same argument applies to the complexification $(\mathrm{d}\varphi)^{\mathbb{C}} : T^{\mathbb{C}}M \to T^{\mathbb{C}}N$. □

4. Harmonic morphisms from Einstein–Weyl 4-manifolds

In this section we shall work in the complex analytic category. We continue the study, initiated in the previous section, of the relation between harmonic morphisms and twistorial maps. We start with a brief presentation of the examples of twistorial maps with which we shall work; more details can be found in [29].

Example 4.1. Let (M^3, c_M, D^M) be a three-dimensional Weyl space and let (N^2, c_N) be a two-dimensional conformal manifold.

A twistorial map $\varphi : (M^3, c_M, D^M) \to (N^2, c_N)$ with nowhere degenerate fibres is a horizontally conformal submersion whose fibres are geodesics with respect to D^M. The existence of such twistorial maps is related to (M^3, c_M, D^M) being Einstein–Weyl [12] (see [29]; see also Remark 4.7(1), below).

Let $\varphi : M^3 \to N^2$ be a submersion with nowhere degenerate fibres and let p, $\widetilde{p}$ be the two-dimensional degenerate distributions locally defined on (M^3, c_M) such that $\ker \mathrm{d}\varphi = p \cap \widetilde{p}$. Then $\varphi : (M^3, c_M) \to (N^2, c_N)$ is horizontally conformal if and only if p and $\widetilde{p}$ are integrable. It follows that $\varphi : (M^3, c_M, D^M) \to (N^2, c_N)$ is twistorial if and only if p and $\widetilde{p}$ are integrable and their integral manifolds are totally-geodesic with respect to D^M; note that φ maps any such surface to a null geodesic on (N^2, c_N).

By Theorems 1.7 and 2.4, $\varphi : (M^3, c_M, D^M) \to (N^2, c_N)$ is a twistorial map if and only if it is a harmonic morphism.

Example 4.2. Let (M^4, c_M) and (N^2, c_N) be oriented conformal manifolds of dimensions 4 and 2, respectively.

A twistorial map $\varphi : (M^4, c_M) \to (N^2, c_N)$ with nowhere degenerate fibres is a horizontally conformal submersion for which the almost complex structure J^M on M^4, with respect to which $\varphi : (M^4, J^M) \to (N^2, J^N)$ is holomorphic and (M^4, c_M, J^M) is a positive almost Hermitian manifold, is integrable (cf. [31]). The existence of such twistorial maps is related to (M^4, c_M) being anti-self-dual (see [22]).

If $\varphi : (M^4, c_M, D^M) \to (N^2, c_N)$ is twistorial and $\mathcal{F}$, $\widetilde{\mathcal{F}}$ are the, necessarily integrable, eigendistributions of J^M then φ maps the leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ to null geodesics on (N^2, c_N).

By Remark 3.3(1), $\varphi : (M^4, c_M) \to (N^2, c_N)$ is twistorial if and only if $D^M J^M = 0$ where D^M is the Weyl connection of (M^4, c_M, J^M). Furthermore, if $\varphi : (M^4, c_M) \to (N^2, c_N)$ is twistorial then, by Proposition 3.5, $\varphi : (M^4, c_M, D^M) \to (N^2, c_N)$ is a harmonic morphism. More generally, if D is a Weyl connection on (M^4, c_M) then, by Proposition 3.7, $\varphi : (M^4, c_M, D) \to (N^2, c_N)$ is twistorial and a harmonic morphism if and only if J^M is parallel along the fibres of φ, with respect to D.

A two-dimensional foliation $\mathcal{V}$ with nowhere degenerate leaves on (M^4, c_M) is twistorial if it can be locally defined by twistorial maps; note that $\mathcal{V}$ is twistorial with respect to both orientations of (M^4, c_M) if and only if its leaves are totally umbilical. If (M^4, c_M) is nonorientable, then $\mathcal{V}$ is twistorial if its lift to the oriented $\mathbb{Z}_2$-covering space of (M^4, c_M) is twistorial; equivalently, $\mathcal{V}$ has totally umbilical leaves.

Example 4.3. Let (M^4, c_M) be an oriented four-dimensional conformal manifold and let (N^3, c_N, D^N) be a three-dimensional Weyl space.

Let $\varphi : (M^4, c_M) \to (N^3, c_N)$ be a horizontally conformal submersion with nowhere degenerate fibres. Let $\mathcal{V} = \ker \mathrm{d}\varphi$, $\mathcal{H} = \mathcal{V}^\perp$ and let D be the Weyl connection of $(M^4, c_M, \mathcal{V})$ (see Example 2.5).

Let g be a positive local representative of the oriented conformal structure c_M, over some open set of M^4, such that $(\mathcal{V}, g|_{\mathcal{V}})$ is orientable. Choose an orientation of $(\mathcal{V}, g|_{\mathcal{V}})$; that is, choose a vertical vector field U such that $g(U,U) = 1$. Orient $(\mathcal{H}, g|_{\mathcal{H}})$ such that the isomorphism of vector bundles $(TM, g) = (\mathcal{V}, g|_{\mathcal{V}}) \oplus (\mathcal{H}, g|_{\mathcal{H}})$ is orientation preserving. Define the horizontal 2-form $I^{\mathcal{H}}$ such that $I^{\mathcal{H}}(X,Y) = -g(U, [X,Y])$ for horizontal X, Y. Let $*_{\mathcal{H}}$ be the Hodge star-operator of $(\mathcal{H}, g|_{\mathcal{H}})$. Then the 1-form $*_{\mathcal{H}} I^{\mathcal{H}}$ is conformally invariant and therefore defines a 1-form on M^4 which depends only of c_M and its orientation.

Let $D_\pm$ be the Weyl partial connections on $(\mathcal{H}, c_M|_{\mathcal{H}})$, over $\mathcal{H}$, given by $\mathcal{H}D \pm *_{\mathcal{H}} I^{\mathcal{H}}$; the map $\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ is twistorial if and only if it is horizontally conformal and the Weyl partial connection on $(\mathcal{H}, c_M|_{\mathcal{H}})$, over $\mathcal{H}$, determined by D^N is equal to D_+ (cf. [5]).

The following assertions are equivalent for a submersion $\varphi : M^4 \to N^3$ with connected nowhere degenerate fibres [5] :

(i) There exists a Weyl connection D^N on (N^3, c_N) with respect to which $\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ is twistorial.

(ii) $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal and the curvature form of the connection induced by D on L_M is anti-self-dual (that is, $\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ is anti-self-dual in the sense of [5]).

If (M^4, c_M) is anti-self-dual then the following assertions can be added to this list [5] (cf. [12] , [14] ; see [29]):

(iii) There exists an Einstein–Weyl connection D^N on (N^3, c_N) such that for any twistorial map ψ locally defined on (N^3, c_N, D^N) with values in a conformal manifold (P^2, c_P) the map $\psi \circ \varphi$ from (M^4, c_M) to (P^2, c_P) is twistorial.

(iv) There exists an Einstein–Weyl connection D^N on (N^3, c_N) such that φ maps self-dual surfaces on (M^4, c_M) to degenerate surfaces on (N^3, c_N) which are totally-geodesic with respect to D^N.

It follows that if $\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ is twistorial then (M^4, c_M) is anti-self-dual if and only if (N^3, c_N, D^N) is Einstein–Weyl [5] (cf. [12], [14]; see [29]).

As in Example 4.2, a one-dimensional foliation $\mathcal{V}$ with nowhere degenerate leaves is twistorial if it can be locally defined by twistorial maps. Note that $\mathcal{V}$ is twistorial with respect to both orientations of (M^4, c_M) if and only if it is locally generated by conformal vector fields [5] (this follows from the equivalence (i) $\iff$ (ii), above, and Example 2.5). If (M^4, c_M) is nonorientable then $\mathcal{V}$ is twistorial if its lift to the oriented $\mathbb{Z}_2$-covering space of (M^4, c_M) is twistorial; equivalently, $\mathcal{V}$ is locally generated by nowhere zero conformal vector fields.

Let D^M and D^N be Weyl connections on (M^4, c_M) and (N^3, c_N), respectively. Let $\varphi : (M^4, c_M) \to (N^3, c_N)$ be a horizontally conformal submersion with nowhere degenerate fibres. From the fundamental equation (2.1) it easily follows that any two of the following assertions imply the third:

(a) $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ is a harmonic morphism.

(b) $\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ is twistorial.

(c) $\mathcal{H}D^M = \mathcal{H}D + \frac{1}{2} *_{\mathcal{H}} I^{\mathcal{H}}$ as partial connections over $\mathcal{H}$.

From Theorem 2.4 it follows that if $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ is a twistorial harmonic morphism with nowhere degenerate fibres then its fibres are geodesics with respect to D^M if and only if $\mathcal{H}$ is integrable; in particular, φ is also twistorial with respect to the reversed orientation of (M^4, c_M), and hence, the fibres of φ are locally generated by nowhere zero conformal vector fields.

The following lemma follows from a straightforward computation.

Lemma 4.4 (cf. [2]). *Let (M, c, D) be a Weyl space, $\dim M = 3, 4$, and let Ric be its Ricci tensor. Let $\mathcal{F}$ be a foliation by null geodesics on (M, c, D) such that $\mathcal{F}^{\perp}$ is integrable.*

(i) *If $\dim M = 3$ then*

$$\mathrm{Ric}(Y, Y) = Y(g(D_U U, Y)) - g(D_U U, Y)^2$$

where $\{U, Y, \widetilde{Y}\}$ is a local frame on M such that Y is a local section of $\mathcal{F}$, $D_Y Y = 0$ and $g = U \odot U + 2\, Y \odot \widetilde{Y}$ is a local representative of c.

(ii) *If* $\dim M = 4$ *then*

$$\mathrm{Ric}(Y,Y) = 2\big[Y(g(D_U\widetilde{U},Y)) + \alpha(Y)g(D_U\widetilde{U},Y) - g([U,Y],U)g([\widetilde{U},Y],\widetilde{U})\big]$$

where $\{U,\widetilde{U},Y,\widetilde{Y}\}$ *is a local frame on* M *such that* Y *is a local section of* $\mathcal{F}$ *,* $D_YY=0$ *,* $g = 2(U\odot\widetilde{U} + Y\odot\widetilde{Y})$ *is a local representative of* c *and* α *is the Lee form of* D *with respect to* g *.* □

Remark 4.5. 1) In Lemma 4.4(i) the condition $\mathcal{F}^\perp$ integrable is superfluous. It follows that from any three-dimensional conformal manifold we can, locally, define horizontally conformal submersions with one-dimensional nowhere degenerate fibres tangent to any given direction at a point. A similar statement holds for real-analytic three-dimensional conformal manifolds.

2) A relation slightly longer than in Lemma 4.4(ii) can be obtained, for a foliation $\mathcal{F}$ by null geodesics on a four-dimensional Weyl space, without the assumption $\mathcal{F}^\perp$ is integrable.

Proposition 4.6 (cf. [2] , [31]). *Let* (M,c_M,D^M) *be a Weyl space, and let* (N,c_N) *be a conformal manifold,* $\dim M = 3,\,4$ *,* $\dim N = 2$ *.*

Let $\varphi : (M,c_M) \to (N,c_N)$ *be a horizontally conformal submersion with nowhere degenerate fibres; suppose* $\mathrm{trace}_{c_M}(D\mathrm{d}\varphi)=0$ *along a hypersurface transversal to the fibres of* φ *.*

Then any two of the following assertions imply the third:

(i) $\varphi : (M,c_M,D^M) \to (N,c_N)$ *is a harmonic morphism.*

(ii) $\varphi : (M,c_M,D^M) \to (N,c_N)$ *is twistorial.*

(iii) *The trace free symmetric part of the horizontal component of the Ricci tensor of* D^M *is zero.*

Proof. If $\dim M = 3$ then (i) $\Longleftrightarrow$ (ii) . Also, we can find a local frame $\{U,Y,\widetilde{Y}\}$, as in Lemma 4.4(i) , such that U is tangent to the fibres of φ . Then assertion (ii) is equivalent to $g(D^M_UU,Y) = g(D^M_UU,\widetilde{Y}) = 0$. On the other hand, assertion (iii) is equivalent to ${}^M\mathrm{Ric}(Y,Y) = {}^M\mathrm{Ric}(\widetilde{Y},\widetilde{Y}) = 0$ where ${}^M\mathrm{Ric}$ is the Ricci tensor of D^M . Thus, if $\dim M = 3$, the proof follows from Lemma 4.4(i) .

Suppose $\dim M = 4$. Then we can find a local frame $\{U,\widetilde{U},Y,\widetilde{Y}\}$ like in Lemma 4.4(ii) , such that U and $\widetilde{U}$ are tangent to the fibres of φ . Moreover, we may assume g oriented such that $\mathcal{F}_+ = Span(U, Y)$ and $\widetilde{\mathcal{F}}_+ = Span(\widetilde{U}, \widetilde{Y})$ are self-dual whilst $\mathcal{F}_- = Span(\widetilde{U}, Y)$ and $\widetilde{\mathcal{F}}_- = Span(U, \widetilde{Y})$ are anti-self-dual. Then assertion (ii) is equivalent to the fact that either $\mathcal{F}_+$, $\widetilde{\mathcal{F}}_+$ are integrable or $\mathcal{F}_-$, $\widetilde{\mathcal{F}}_-$ are integrable. On the other hand, assertion (i) is equivalent to $g(D^M_U \widetilde{U}, Y) = g(D^M_U \widetilde{U}, \widetilde{Y}) = 0$ (see the proof of Proposition 3.7). Thus, if $\dim M = 4$, the proof follows from Lemma 4.4(ii). □

Remark 4.7. 1) If $\dim M = 3$ then assertions (i), (ii), (iii) of Proposition 4.6 are equivalent.

It follows that if (M, c_M, D^M) is a three-dimensional Weyl space from which can be locally defined more than $k = 6$ harmonic morphisms with one-dimensional nowhere degenerate fibres then (M, c_M, D^M) is Einstein–Weyl; in the smooth category, the same statement holds with $k = 2$ [6] (cf. [2]).

2) In the smooth category, suppose assertion (i) of Proposition 4.6 holds.

(a) If $\dim M = 3$ then assertion (iii) also holds (cf. [2]).

(b) If $\dim M = 4$ then the implication (ii)⇒(iii) holds on M whilst the implication (iii)⇒(ii) holds locally on a dense open set of M (cf. [31]).

3) Proposition 4.6 also holds for any horizontally conformal submersion $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ with nowhere degenerate fibres from a Weyl space to an Einstein–Weyl space, $\dim M = 4$, $\dim N = 3$ (see Proposition 4.9, below, for an extension of this fact).

The proof of the following lemma is omitted.

Lemma 4.8 (cf. [28]). *Let $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ be a submersive harmonic morphism with nowhere degenerate fibres between Weyl spaces,* $\dim M = 4$, $\dim N = 3$.

Let $A_\pm = D_\pm - D^N$. Then for any horizontal null vector Y we have

$$^M\mathrm{Ric}(Y, Y) - {}^N\mathrm{Ric}(\mathrm{d}\varphi(Y), \mathrm{d}\varphi(Y)) = -\tfrac{1}{2}\, A_+(Y)\, A_-(Y)$$

where $^M\mathrm{Ric}$ and $^N\mathrm{Ric}$ are the Ricci tensors of D^M and D^N, respectively. □

The following result follows from Lemmas 4.4(ii) and 4.8.

Proposition 4.9 (cf. [28]). *Let $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ be a nonconstant harmonic morphism with nowhere degenerate fibres between Weyl spaces,* $\dim M = 4$, $\dim N = 2, 3$. *Let $^M\mathrm{Ric}$ and $^N\mathrm{Ric}$ be the Ricci tensors of D^M and*

D^N , respectively.

Then the following assertions are equivalent:

(i) *φ is twistorial.*

(ii) *The trace free symmetric part of the horizontal component of* ${}^M\mathrm{Ric} - \varphi^*({}^N\mathrm{Ric})$ *is zero.* □

From Examples 4.2 , 4.3 and Proposition 4.9 we obtain the following two results.

Theorem 4.10 (cf. [31]). *Let (M^4, c_M, D^M) be an Einstein–Weyl space of dimension four and let $\varphi : (M^4, c_M, D^M) \to (N^2, c_N)$ be a submersive harmonic morphism with nowhere degenerate fibres to a conformal manifold of dimension two.*

If (M^4, c_M) is orientable then, with respect to a suitable orientation, φ is twistorial.

If (M^4, c_M) is nonorientable then the fibres of φ are totally umbilical. □

Theorem 4.11 (cf. [26] , [28] , [29]). *Let (M^4, c_M, D^M) and (N^3, c_N, D^N) be Einstein–Weyl spaces of dimension four and three, respectively.*

Let $\varphi : (M^4, c_M, D^M) \to (N^3, c_N)$ be a submersive harmonic morphism with nowhere degenerate fibres.

If (M^4, c_M) is orientable then it is anti-self-dual, with respect to a suitable orientation, and φ is twistorial.

If (M^4, c_M) is nonorientable then it is conformally flat, the horizontal distribution of φ is integrable and the fibres of φ are locally generated by conformal vector fields whose orbits are geodesics with respect to D^M . □

Remark 4.12. 1) Theorems 4.10 and 4.11 also hold in the smooth category.

2) Further results can be obtained for a submersive harmonic morphism $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ between Einstein-Weyl spaces by combining Theorem 4.11 with known facts from four-dimensional Weyl geometry.

If (M^4, c_M) is orientable, as (M^4, c_M, D^M) is Einstein–Weyl and anti-self-dual with respect to a suitable orientation, we have that, locally, either D^M is the Obata connection of a hyper-Hermitian structure on (M^4, c_M) or D^M is the Levi-Civita connection of an Einstein representative of c_M (see [5]). As φ is a twistorial harmonic morphism, from Proposition 3.7 and [5] it follows that, locally, one of the following assertions holds:

(i) (N^3, c_N, D^N) is Gauduchon-Tod [9] and D^M is the Obata connection of the induced hyper-Hermitian structure on (M^4, c_M).

(ii) D^M is the Levi-Civita connection of an Einstein representative g of c_M with respect to which (M^4, g) is the $\mathcal{H}$-space [18] of (N^3, c_N) and φ is the retraction of $N^3 \hookrightarrow M^4$ corresponding to D^N [12].

If (M^4, c_M) is nonorientable it follows that φ is a harmonic morphism of warped product type between Riemannian manifolds of constant curvature.

The harmonic morphisms given by the Gibbons-Hawking and the Beltrami fields constructions (see [28]) satisfy assertion (i) whilst the harmonic morphisms of warped product type (see [3]) with one-dimensional fibres from a four-dimensional Riemannian manifold with nonzero constant sectional curvature satisfy assertion (ii).

References

[1] P. Baird, J. Eells, A conservation law for harmonic maps, *Geometry Symposium. Utrecht 1980*, 1–25, Lecture Notes in Math. 894, Springer-Verlag, Berlin, 1981.

[2] P. Baird, J.C. Wood, Harmonic morphisms, Seifert fibre spaces and conformal foliations, *Proc. London Math. Soc.*, **64** (1992) 170–196.

[3] P. Baird, J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.), no. 29, Oxford Univ. Press, Oxford, 2003.

[4] R. Bott, Lectures on characteristic classes and foliations. Notes by Lawrence Conlon, with two appendices by J. Stasheff, *Lectures on algebraic and differential topology (Second Latin American School in Math., Mexico City, 1971)*, 1–94. Lecture Notes in Math. 279, Springer, Berlin, 1972.

[5] D.M.J. Calderbank, Selfdual Einstein metrics and conformal submersions, *Preprint*, Edinburgh University, 2000.

[6] D.M.J. Calderbank, H. Pedersen, Selfdual spaces with complex structures, Einstein–Weyl geometry and geodesics, *Ann. Inst. Fourier (Grenoble)*, **50** (2000) 921–963.

[7] B. Fuglede, Harmonic morphisms between Riemannian manifolds, *Ann. Inst. Fourier (Grenoble)*, **28** (1978) 107–144.

[8] P. Gauduchon, Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $\mathbf{S}^1 \times \mathbf{S}^3$, *J. Reine Angew. Math.*, **469** (1995) 1–50.

[9] P. Gauduchon, K.P. Tod, Hyper-Hermitian metrics with symmetry, *J. Geom. Phys.*, **25** (1998) 291–304.

[10] S. Gudmundsson, *The Bibliography of Harmonic Morphisms*, `http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html`

[11] S. Gudmundsson, J.C. Wood, Harmonic morphisms between almost Hermitian manifolds, *Boll. Unione Mat. Ital.* (7), **11-B** (1997), Supplement to fasc. 1, 30–58.

[12] N.J. Hitchin, Complex manifolds and Einstein's equations, *Twistor geometry and nonlinear systems (Primorsko, 1980)*, 73–99, Lecture Notes in Math., 970, Springer, Berlin, 1982.

[13] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, *J. Math. Kyoto Univ.*, **19** (1979) 215–229.

[14] P.E. Jones, K.P. Tod, Minitwistor spaces and Einstein–Weyl spaces, *Classical Quantum Gravity*, **2** (1985) 565–577.

[15] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, I, II, Wiley Classics Library (reprint of the 1963, 1969 original), Wiley-Interscience Publ., Wiley, New-York, 1996.

[16] K. Kodaira, A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, *Ann. of Math.*, **75** (1962) 146–162.

[17] C.R. LeBrun, Spaces of complex geodesics and related structures, Ph.D. Thesis, Oxford, 1980.

[18] C.R. LeBrun, $\mathcal{H}$-space with a cosmological constant, *Proc. Roy. Soc. London Ser. A*, **380** (1982) 171–185.

[19] C.R. LeBrun, Spaces of complex null geodesics in complex-Riemannian geometry, *Trans. Amer. Math. Soc.*, **278** (1983) 209–231.

[20] A. Lichnerowicz, Applications harmoniques et variétés kähleriennes, *Symposia Mathematica, Vol. III (INDAM, Rome, 1968/1969)*, 341–402.

[21] E. Loubeau, Hermitian harmonic maps, *Beiträge Algebra Geom.*, **40** (1999) 1–14.

[22] L.J. Mason, N.M.J. Woodhouse, *Integrability, self-duality, and twistor theory*, London Math. Soc. Monogr. (N.S.), no. 15, Oxford Univ. Press., Oxford, 1996.

[23] A. Pambira, Harmonic morphisms between degenerate semi-Riemannian manifolds, Preprint, University of Leeds, 2003.

[24] R. Pantilie, Harmonic morphisms with one-dimensional fibres, *Internat. J. Math.*, **10** (1999) 457–501.

[25] R. Pantilie, Submersive harmonic maps and morphisms, Ph.D. Thesis, University of Leeds, 2000.

[26] R. Pantilie, Harmonic morphisms with 1-dimensional fibres on 4-dimensional Einstein manifolds, *Comm. Anal. Geom.*, **10** (2002) 779–814.

[27] R. Pantilie, J.C. Wood, Harmonic morphisms with one-dimensional fibres on Einstein manifolds, *Trans. Amer. Math. Soc*, **354** (2002) 4229–4243.

[28] R. Pantilie, J.C. Wood, A new construction of Einstein self-dual manifolds, *Asian J. Math.*, **6** (2002) 337–348.

[29] R. Pantilie, J.C. Wood, Twistorial harmonic morphisms with one-dimensional fibres on self-dual four-manifolds, Preprint, University of Leeds, 2003.

[30] I. Vaisman, On locally conformal almost Kähler manifolds, *Israel J. Math*, **24** (1976) 338–351.

[31] J.C. Wood, Harmonic morphisms and Hermitian structures on Einstein 4-manifolds, *Internat. J. Math.*, **3** (1992) 415–439.

E-mail address: `Eric.Loubeau@univ-brest.fr,Radu.Pantilie@univ-brest.fr`

Département de Mathématiques, Laboratoire C.N.R.S. U.M.R. 6205, Université de Bretagne Occidentale, 6, Avenue Victor Le Gorgeu, CS 93837, 29238 Brest Cedex 3, France