Transient random walk in $\mathbb{Z}^2$ with stationary orientations

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Abstract. *In this paper, we extend a result of Campanino and Petritis [1]. We study a random walk in $\mathbb{Z}^2$ with a random environment. We suppose that the orientations of the horizontal floors are given by a stationary sequence of random variables $(\xi_k)_{k\in\mathbb{Z}}$. Once the environment fixed, the random walk can go either up or down or with respect to the orientation of the present floor (with the same probability). In [1], the $(\xi_k)_{k\in\mathbb{Z}}$ is a sequence of independent identically distributed random variables. In [2], this result is extended to some cases of independent orientations choosen with stationary probabilities (not equal to 0 and to 1). In the present paper, we generalize the result of [1] to some cases when $(\xi_k)_k$ is stationary. Moreover we give a slight extension of a result of [2].*

1 Introduction

We consider a random walk $(M_n = (\tilde{X}_n, \tilde{Y}_n))_n$ in $\mathbb{Z}^2$ with a random environment of a specific type. We suppose that $M_0 = (0,0)$. Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary sequence of centered random variables with values in $\{-1;1\}$. The orientations of the k^{th} horizontal floor of $\mathbb{Z}^2$ is given by ξ_k. Once the environment fixed, the random walk $(M_n = (\tilde{X}_n, \tilde{Y}_n))_n$ will be such that the distribution of $M_{n+1} - M_n$ conditionned to $\sigma(M_k; k = 0, ..., n)$ is uniform on $\{(0,1); (0,-1); (\xi_{\tilde{Y}_n}, 0\}$.

In [1], Campanino and Petritis prove the transience of the random walk $(M_n)_n$ when $(\xi_k)_{k\in\mathbb{Z}}$ is sequence of independent identically distributed random variables.

In [2], the following situation is envisaged : Let $(f_k)_{k\in\mathbb{Z}}$ be a stationary sequence of random variables with values in $[0;1]$ and with expectation equal to $\frac{1}{2}$ defined on some probability space $(M, \mathcal{F}, \nu)$. Let us consider the probability space given by $(\Omega_1 := M \times]0;1[^{\mathbb{Z}}, \mathcal{F}_2 := \mathcal{F} \otimes (\mathcal{B}(]0;1[))^{\otimes\mathbb{Z}}, \nu_1 := \nu \otimes (\lambda)^{\otimes\mathbb{Z}})$, where λ is the Lebesgue measure on $]0;1[$. We define $(\tilde{\xi}_{k,f_k})_{k\in\mathbb{Z}}$ on this space as follows :

$$\tilde{\xi}_{k,f_k}(\omega, (z_m)_{m\in\mathbb{Z}}) := 2.\mathbf{1}_{\{z_k \le f_k(\omega)\}} - 1.$$

This means that, once a realization of $(f_k)_k$ given, the horizontal floors are oriented independently; the k^{th} floor being oriented to the right with probability f_k. In [2], Guillotin and Le Ny prove that, if $(\xi_k)_k = \left(\tilde{\xi}_{k,f_k}\right)_k$, then the corresponding random walk $(M_n)_n$ is transient under the following condition : $\int_M \frac{1}{\sqrt{f(1-f)}}\, d\nu < +\infty$.

Let us notice that the $(\xi_k)_k$ studied in [2] is stationary. Let us notice that, conversely, if $(\xi_k)_k$ is stationary, then it can be described by the approach of [2] by taking $f_k := \mathbf{1}_{\{\xi_k=1\}} = \frac{1}{2}(\xi_k + 1)$. But the method of [2] cannot be applied to a function f that can be equal to 0 or 1 with a non-null probability.

In this paper, we are interested in the case when $(\xi_k)_{k\in\mathbb{Z}}$ is a stationary sequence of random variables satisfying some strong decorrelation properties.

We also end this paper with a short discussion about the model envisaged by Guillotin and Le Ny. We prove that their result remains true if the condition $\int_M \frac{1}{\sqrt{f(1-f)}}\,d\nu < +\infty$ is replaced by $\int_M \frac{1}{[f(1-f)]^p}\,d\nu < +\infty$, for some $p > 0$.

2 Result

Theorem 1 *Let us suppose that :*

1. *we have :*
$$\sum_{p\geq 0}\sqrt{1+p}\,|\mathbb{E}[\xi_0\xi_p]| < +\infty$$
$$\text{and}\quad c_0' := \sup_{N\geq 1} N^{-2}\sum_{k_1,k_2,k_3,k_4=0,\dots,N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]| < +\infty.$$

2. *There exists some $C>0$, some $(\varphi_{p,s})_{p,s\in\mathbb{N}}$ and some integer $r\geq 1$ such that*
$$\forall (p,s)\in\mathbb{N}^2,\quad \varphi_{p+1,s}\leq\varphi_{p,s}\quad \text{and}\quad \lim_{s\to+\infty} s^6\varphi_{rs,s}=0$$
and such that, for all integers n_1,n_2,n_3,n_4 with $0\leq n_1\leq n_2\leq n_3\leq n_4$, for all real numbers $\alpha_{n_1},\dots,\alpha_{n_2}$ and $\beta_{n_3},\dots,\beta_{n_4}$, we have :
$$\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\xi_k}, e^{i\sum_{k=n_3}^{n_4}\beta_k\xi_k}\right)\right| \leq C\left(1+\sum_{k=n_1}^{n_2}|\alpha_k|+\sum_{k=n_3}^{n_4}|\beta_k|\right)\varphi_{n_3-n_2,n_4-n_3}.$$

Then the random walk $(M_n)_n$ is transient.

Let us notice that the hypotheses of this theorem are satisfied under the following α-mixing condition :
$$\lim_{n\to+\infty} n^6 \sup_{p\geq 0; m\geq 0}\ \sup_{A\in\sigma(\xi_{-p},\dots,\xi_0)}\ \sup_{B\in\sigma(\xi_n,\dots,\xi_{n+m})} |\mathbb{P}(A\cap B)-\mathbb{P}(A)\mathbb{P}(B)| = 0.$$

Moreover, in [4], we give examples of dynamical systems $(M,\mathcal{F},\nu,T)$ and of class of functions $\mathcal{C}$ such that :

- If $f : M\to[0;1]$ belongs to the class $\mathcal{C}$ and if $\int_M f\,d\nu = \frac{1}{2}$ then $(\xi_k := \tilde{\xi}_{k,f\circ T^k})_{k\in\mathbb{Z}}$ satisfies the hypotheses of our theorem 1.
- If $g : M\to\{\pm 1\}$ belongs to the class $\mathcal{C}$ and if $\int_M g\,d\nu = 0$, then $(\xi_k := g\circ T^k)_{k\in\mathbb{Z}}$ satisfies the hypotheses of our theorem 1.

3 Proof of theorem 1

Let us define $T_0 := 0$ and, for all $n\geq 1$: $T_{n+1} := \inf\{k > T_n \ :\ \tilde{Y}_k \neq \tilde{Y}_{k-1}\}$. According to lemma 2.5 of [1], we have the following result :

Lemma 2 *If $(M_{T_n})_{n\geq 0}$ is transient, then $(M_n)_{n\geq 0}$ is transient*

Now, still following [1], we construct a realization of $(M_{T_n})_n$:

Let us consider a symmetric random walk $(S_n)_n$ on $\mathbb{Z}$ independent of $(\xi_k)_{k\in\mathbb{Z}}$. For any integer $m\geq 1$ and any integer k, we define :
$$N_m(k) := Card\{j=0,\dots,m \ :\ S_j = k\}.$$

Let us also consider a sequence of independent random variables $(\zeta_i^{(y)})_{i\geq 1, y\in\mathbb{Z}}$ with geometric distribution with parameter $\frac{1}{3}$, and independent of $(\xi_y)_{y\in\mathbb{Z}}$.

Lemma 3 *The process* $\left(X_n := \sum_{y\in\mathbb{Z}} \xi_y \sum_{i=1}^{N_{n-1}(y)} \xi_i^{(y)}, S_n\right)_{n\geq 1}$ *has the same distribution as* $(M_{T_n})_n$.

In this lemma, $\xi_i^{(y)}$ corresponds to the duration of the stay at the y^{th} horizontal floor during the i^{th} visit to this floor.

According to the Borel-Cantelli lemma, it suffices to prove that : $\sum_{n\geq 1} \mathbb{P}(\{(X_n, S_n) = (0,0)\}) < +\infty$.

We follow the scheme of the proof of [1]. The difference will be in our way to estimate $I_n^{(1)}$ and in the introduction of the sets U_n. We will consider δ_1, δ_2, δ_3, and γ such that :

$0 < \delta_1 < 2\delta_2$, $\delta_1 + (\frac{27}{2} + 16)\delta_2 < \frac{1}{8}$, $\delta_3 > 0$, $\frac{1}{4} - 3\delta_2 < \delta_3 < \frac{1}{4} - \frac{5}{2}\delta_2 - \delta_1$, $\frac{\delta_3}{2} - 2\delta_2 < \beta < \frac{\delta_3}{2} - \delta_2$, $\max(\delta_1, \delta_2) < \gamma < \frac{1}{2} - 22\max(\delta_1, \delta_2)$. The idea is that δ_1, δ_2, $\frac{1}{4} - \delta_3$ and $\frac{1}{8} - \beta$ are positive numbers very close to zero.

As in [1, 2], let us define :

$$A_n := \{\omega \in \Omega \ : \ \max_{\ell\in\mathbb{Z}} N_{n-1}(\ell) \leq n^{\frac{1}{2}+\delta_2} \text{ and } \max_{k=0,\dots,n} |S_k| < n^{\frac{1}{2}+\delta_1}\}.$$

Moreover, we define :

$$U_n := \{\omega \in A_n \ : \ \forall x, y \in \mathbb{Z},\ |N_{n-1}(x) - N_{n-1}(y)| \geq \sqrt{|x-y| n^{\frac{1}{2}+\gamma}}\}.$$

The sketch of the proof is the following :

1. As in proposition 4.1 of [1], we have :

$$\sum_{n\geq 1} \mathbb{P}\left(\{X_n = 0 \text{ and } S_n = 0\} \setminus A_n\right) < +\infty;$$

   actually we have : $\sum_{n\geq 1} \mathbb{P}\left(\{S_n = 0\} \setminus A_n\right) < +\infty$;

2. We will see in lemma 4 of the present paper that we have :

$$\sum_{n\geq 1} \mathbb{P}\left(A_n \setminus U_n\right) < +\infty.$$

   Therefore, we have :

$$\sum_{n\geq 0} \mathbb{P}\left(\{X_n = 0 \text{ and } S_n = 0\} \setminus U_n\right) < +\infty;$$

3. Let us define $B_n := \{\omega \in U_n \ : \ \left|\sum_{y\in\mathbb{Z}} \xi_y N_{n-1}(y)\right| > n^{\frac{1}{2}+\delta_3}\}$. As in proposition 4.3 of [1],we have :

$$\sum_{y\in\mathbb{Z}} \mathbb{P}(B_n \cap \{X_n = 0 \text{ and } S_n = 0\}) < +\infty.$$

   It remains to prove that :

$$\sum_{n\geq 0} \mathbb{P}\left(U_n \cap \{X_n = 0 \text{ and } S_n = 0\} \setminus B_n\right) < +\infty.$$

   (a) As in lemma 4.5 of [1], there exists a real number $C > 0$ such that :

$$\sup_{\omega\in U_n\setminus B_n} \mathbb{P}\left(\{X_n = 0\} | (S_p)_{p\geq 1}, (\xi_k)_{k\in\mathbb{Z}}\right) \leq C\sqrt{\frac{\ln(n)}{n}}.$$

(b) We will prove that there exists some $\tilde\delta > 0$ and some $C' > 0$ such that :

$$\forall \omega \in U_n,\quad \mathbb{P}\left(U_n \setminus B_n | (S_p)_p\right)(\omega) \le C' n^{-\tilde\delta}.$$

i. This probability is bounded by $c' n^{\frac12+\delta_3} I_n(\omega)$ with $I_n(\omega) = I_n^{(1)}(\omega) + I_n^{(2)}(\omega)$ and

$$I_n^{(1)}(\omega) := \int_{\{|t|\le n^{-\frac12-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}} \xi_y N_{n-1}(y)(\omega)} \Big| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt$$

and

$$I_n^{(2)}(\omega) := \int_{\{|t|> n^{-\frac12-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}} N_{n-1}(y)(\omega)} \Big| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt.$$

ii. We will prove that $n^{\frac12+\delta_3} I_n^{(1)} = O(n^{-\delta})$ for some $\delta > 0$ (see our lemma 5);

iii. On the other hand, following [1], we have :

$$n^{\frac12+\delta_3} I_n^{(2)} \le \int_{\{|s|>n^{\delta_2}\}} e^{-\frac{s^2}{2}}\, ds \le 2n^{-\delta_2} e^{-\frac{n^2\delta_2}{2}}$$

(c) We have $\mathbb{P}(Y_{n+1} = 0) \le C'' n^{-\frac12}$.

(d) Hence we have : $\mathbb{P}\left(U_n \cap \{X_n = 0 \text{ and } S_n = 0\} \setminus B_n\right) \le C''' n^{-1-\delta}\sqrt{\ln(n)}$.

We have to prove that the points 2 and 3(b)(ii) are true with our choices of parameters. All the other points are true for any positive $\delta_1, \delta_2, \delta_3$ and for any sequence of random variables $(\xi_k)_{k\in\mathbb{Z}}$ independent of $(S_p)_p$.

We notice that, for any integer $n \ge 1$, we have : $\sum_{j=0}^{n-1} \xi_{S_j} = \sum_{k\in\mathbb{Z}} \xi_k N_{n-1}(k)$. In our proof, we need some real numbers δ_1, δ_2, δ_3, δ_4, β, γ and $\varepsilon > 0$. We will suppose that :

$\delta_1 > 0$, $\delta_2 > 0$, $\delta_1 + (\frac{27}{2} + 16)\delta_2 < \frac18$, $\delta_3 > 0$, $\delta_1 < \delta_4 < \frac14 - \delta_3 - \frac52\delta_2$, $\frac14 - 3\delta_2 < \delta_3 < \frac14 - \frac52\delta_2$, $\frac53\delta_2 < \frac12\delta_3$, $\frac{\delta_3}{2} - 2\delta_2 < \beta < \frac{\delta_3}{2} - \delta_2$, $\frac52\delta_3 > \frac12 + 6\delta_2 + \delta_1$, $\max(\delta_1, \delta_2) < \gamma < \frac12 - 22\max(\delta_1,\delta_2)$ and :

$$n^{\delta_1+11\delta_2} \sum_{m \ge \frac{(r+1)n^\beta}{2}} |\mathbb{E}[\xi_0\xi_m]| = O(n^{-\varepsilon}).$$

(let us recall that we have supposed : $\sum_{m\ge N} |\mathbb{E}[\xi_0\xi_m]| \le N^{-\frac12} \sum_{m\ge N} \sqrt{m}|\mathbb{E}[\xi_0\xi_m]|$)

All this inequalities are true with the following choices of parameters :

$$\delta_1 = \frac{1}{3000},\ \delta_2 = \frac{1}{500},\ \delta_3 = \frac14 - \frac{11}{4}\delta_2,\ \delta_4 = 1/2500,\ \beta = \frac{\delta_3}{2} - \frac32\delta_2 = \frac18 - \frac{23}{8}\delta_2,\ \gamma = \frac14.$$

Lemma 4 *We have :*

$$\sum_{n\ge1} \mathbb{P}\left(A_n \setminus U_n\right) < +\infty.$$

Proof. Let us consider any $x, y \in \mathbb{Z}$ with $x \ne y$ and $|x-y| \le 3n^{\frac12+\delta_1}$. For any integer $j \ge 1$, we define $\tau_j(x)$ the time of the j^{th} visit of $(S_p)_p$ to x and $M_j(x,y)$ the number of visits of $(S_p)_p$ to y between the times $\tau_j(x)$ and $\tau_{j+1}(x)$. According to [3, 5], for any integer $p \ge 1$, there exists $K_p > 0$ such that, for any $x', y,'$ we have :

$$\mathbb{E}[(M_j(x',y'))^p] \le K_p |x'-y'|^{2p-1}.$$

According to [3], on the set $\{\tau_1(x) \le \tau_1(y)\}$, we have :

$$(N_{n-1}(x) - N_{n-1}(y)) = \sum_{j=1}^{N_{n-1}(x)} (1 - M_j(x,y)) + \sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)} \mathbf{1}_{\{S_k = y\}}.$$

Let p be any positive integer. We have :

$$(N_{n-1}(x)-N_{n-1}(y))^{2p}\mathbf{1}_{\{\tau_1(x)\le\tau_1(y)\}}\le 2^{2p}\left[\left(\sum_{j=1}^{N_{n-1}(x)}(1-M_j(x,y))\right)^{2p}+\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)}\mathbf{1}_{\{S_k=y\}}\right)^{2p}\right].$$

But, on A_n, since we have $N_{n-1}(x)\le n^{\frac{1}{2}+\delta_2}$, we get :

$$\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)}\mathbf{1}_{\{S_k=y\}}\right)^{2p}\le\left(M_{N_{n-1}(x)}(x,y)\right)^{2p}\le\sum_{j=1}^{\lfloor n^{\frac{1}{2}+\delta_2}\rfloor}(M_j(x,y))^{2p}.$$

Hence we have:

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)}\mathbf{1}_{\{S_k=y\}}\right)^{2p}\mathbf{1}_{A_n}\right] &\le n^{\frac{1}{2}+\delta_2}K_{2p}|x-y|^{2p-1}\\ &\le n^{\frac{1}{2}+\delta_2}K_{2p}|x-y|^{p}\left(3n^{\frac{1}{2}+\delta_1}\right)^{p-1}\\ &\le K_{2p}3^{p-1}|x-y|^p\left(n^{\frac{1}{2}+\max(\delta_1,\delta_2)}\right)^p.\end{aligned}$$

Moreover, on A_n, we have :

$$\left(\sum_{j=1}^{N_{n-1}(x)}(1-M_j(x,y))\right)^{2p}\le\max_{k=1,\dots,\lfloor n^{\frac{1}{2}+\delta_2}\rfloor}\left(\sum_{j=1}^{k}(1-M_j(x,y))\right)^{2p}.$$

Since $\left(\sum_{j=1}^k(1-M_j(x,y))\right)_{k\ge1}$ is a martingale, according to a maximal inequality, we have :

$$\left\|\max_{k=1,\dots,\lfloor n^{\frac{1}{2}+\delta_2}\rfloor}\left(\sum_{j=1}^{k}(1-M_j(x,y))\right)^{2}\right\|_{L^p}\le\frac{p}{p-1}\max_{k=1,\dots,\lfloor n^{\frac{1}{2}+\delta_2}\rfloor}\left\|\left(\sum_{j=1}^{k}(1-M_j(x,y))\right)^{2}\right\|_{L^p}.$$

Hence we have :

$$\mathbb{E}\left[\left(\sum_{j=1}^{N_{n-1}(x)}(1-M_j(x,y))\right)^{2p}\mathbf{1}_{A_n}\right]\le\left(\frac{p}{p-1}\right)^p\max_{k=1,\dots,\lfloor n^{\frac{1}{2}+\delta_2}\rfloor}\mathbb{E}\left[\left(\sum_{j=1}^{k}(1-M_j(x,y))\right)^{2p}\right].$$

For any $k=1,\dots,\lfloor n^{\frac{1}{2}+\delta_2}\rfloor$, we have :

$$\mathbb{E}\left[\left(\sum_{j=1}^{k}(1-M_j(x,y))\right)^{2p}\right]=\sum_{l=1}^{2p}\sum_{\nu_1+\dots+\nu_l=2p;\ \min_i\nu_i\ge1}\mathcal{M}^{2p}_{\nu_1,\dots,\nu_l}\sum_{j_1<\dots<j_l}\prod_{m=1}^{l}\mathbb{E}[(1-M_{j_m}(x,y))^{\nu_m}],$$

(since the M_{j_m} are independent) with $\mathcal{M}^{2p}_{\nu_1,\dots,\nu_l}=\frac{(2p)!}{\prod_{i=1}^l\nu_i!}$. Since $\mathbb{E}[1-M_j(x,y)]=0$, we have :

$$\mathbb{E}\left[\left(\sum_{j=1}^{k}(1-M_j(x,y))\right)^{2p}\right]\le$$

$$
\begin{aligned}
&\le \sum_{l=1}^{2p} \sum_{\nu_1+\ldots+\nu_l=2p;\ \min_i \nu_i\ge 2} \mathcal{M}^{2p}_{\nu_1,\ldots,\nu_l} \sum_{j_1<\ldots<j_l} \prod_{m=1}^{l} \left(2^{\nu_m}\mathbb{E}[1+(M_{j_m}(x,y))^{\nu_m}]\right)\\
&\le \sum_{l=1}^{2p} \sum_{\nu_1+\ldots+\nu_l=2p;\ \min_i \nu_i\ge 2} \mathcal{M}^{2p}_{\nu_1,\ldots,\nu_l} \sum_{j_1<\ldots<j_l} \prod_{m=1}^{l} \left(2^{\nu_m}(1+K_{\nu_m}|x-y|^{\nu_m-1}\right)\\
&\le 2^{2p}\sum_{l=1}^{2p}|x-y|^{2p-l}(n^{\frac12+\delta_2})^l \sum_{\nu_1+\ldots+\nu_l=2p;\ \min_i \nu_i\ge 2} \mathcal{M}^{2p}_{\nu_1,\ldots,\nu_l}\prod_{m=1}^{l}(1+K_{\nu_m})\\
&\le \tilde{C}_p\sum_{l=1}^{2p}|x-y|^{2p-l}(n^{\frac12+\delta_2})^l\\
&\le \tilde{C}_p\sum_{l=1}^{2p}|x-y|^{p}(3n^{\frac12+\delta_1})^{p-l}(n^{\frac12+\delta_2})^l\\
&\le 2p3^p\tilde{C}_p|x-y|^p(n^{\frac12+\max(\delta_1,\delta_2)})^p.
\end{aligned}
$$

Hence we get :

$$\mathbb{E}\left[(N_{n-1}(x)-N_{n-1}(y))^{2p}\mathbf{1}_{A_n}\right]\le \tilde{C}'_p|x-y|^p(n^{\frac12+\max(\delta_1,\delta_2)})^p.$$

Therefore, according to the Markov inequality, for any integer $p\ge 1$, we have :

$$
\begin{aligned}
\mathbb{P}\left(A_n\setminus U_n\right) &\le \sum_{x,y=-\lceil n^{\frac12+\delta_1}\rceil}^{\lceil n^{\frac12+\delta_1}\rceil} \mathbb{P}\left(A_n\cap\left\{|N_{n-1}(x)-N_{n-1}(y)|>\sqrt{|x-y|n^{\frac12+\gamma}}\right\}\right)\\
&\le \sum_{x,y=-\lceil n^{\frac12+\delta_1}\rceil}^{\lceil n^{\frac12+\delta_1}\rceil} \frac{\mathbb{E}[(N_{n-1}(x)-N_{n-1}(y))^{2p}\mathbf{1}_{A_n}]}{|x-y|^p(n^{\frac12+\gamma})^p}\\
&\le c_p\left(5n^{\frac12+\delta_1}\right)^2\left(n^{\max(\delta_1,\delta_2)-\gamma}\right)^p.
\end{aligned}
$$

By taking p large enough, we get : $\sum_{n\ge1}\mathbb{P}(A_n\setminus U_n)<+\infty$, *qed.*

3.1 Estimates on U_n

In this section, we suppose that we are in U_n. We will estimate :

$$I_n^{(1)}(\omega):=\int_{\{|t|\le n^{-\frac12-\delta_3+\delta_2}\}}\left(\mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}}\xi_yN_{n-1}(y)}\Big|(S_p)_p\right](\omega)\right)e^{-\frac{t^2n^{1+2\delta_3}}{2}}\,dt.$$

Lemma 5 *There exists some real number $\delta>0$ such that we have :*

$$\sup_{n\ge1}n^\delta\sup_{\omega\in U_n}n^{\frac12+\delta_3}I_n^{(1)}(\omega)<+\infty.$$

We will use the following formula :

$$n^{\frac12+\delta_3}I_n^{(1)}(\omega)=n^{\delta_2}\int_{\{|u|\le1\}}\left(\mathbb{E}\left[e^{iun^{-\frac12-\delta_3+\delta_2}\sum_{y\in\mathbb{Z}}\xi_yN_{n-1}(y)}\Big|(S_p)_p\right](\omega)\right)e^{-\frac{u^2n^{2\delta_2}}{2}}\,du.$$

The main idea is to prove that, in $I_n^{(1)}$, we can replace the term :

$$B_n(u)(\omega):=\mathbb{E}\left[e^{iun^{-\frac12-\delta_3+\delta_2}\sum_{y\in\mathbb{Z}}\xi_yN_{n-1}(y)}\Big|(S_p)_p\right](\omega)$$

by

$$A_n(u)(\omega) := \exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_{y,z}\mathbb{E}[\xi_y\xi_z](N_{n-1}(y)(\omega))^2\right).$$

More precisely we will prove the following :

Lemma 6 *There exists a real number $\delta_0 > 0$ such that we have :*

$$\sup_{n\geq 1} n^{\delta_0}\sup_{\omega\in U_n} n^{\delta_2}\int_{|u|\leq 1}|B_n(u)(\omega)-A_n(u)(\omega)|\, e^{-\frac{u^2n^{2\delta_2}}{2}}\,du < +\infty. \qquad (1)$$

Lemma 5 will be an easy consequence of lemma 6.

We will use the following notation : $\sigma_\xi^2 := \sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]$.

3.1.1 Proof of lemma 6

We adapt the idea used in [4] to establish a result of convergence in distribution for $\left(\sum_{k=0}^{n-1}\xi_{S_n} = \sum_y \xi_y N_{n-1}(y)\right)_{n\geq 1}$. Let n be an integer such that $n^\beta \geq 2$. Let us fix $\omega\in U_n$ and $u\in[-1;1]$. Let us recall that $0<\beta<\frac{\delta_3}{2}-\delta_2$ Let us define :

$$L_n := \left\lfloor \frac{2\left\lfloor n^{\frac{1}{2}+\delta_1}\right\rfloor + 1}{\lfloor n^\beta\rfloor}\right\rfloor.$$

We notice that we have : $L_n \leq 4n^{\frac{1}{2}+\delta_1-\beta}$ since $\lfloor n^\beta\rfloor \geq \frac{n^\beta}{2}$.

$$\forall k=0,...,L_n,\quad \alpha_{(k)} := -\left\lfloor n^{\frac{1}{2}+\delta_1}\right\rfloor + k\lfloor n^\beta\rfloor \text{ and } \alpha_{(L_n+1)} := \left\lfloor n^{\frac{1}{2}+\delta_1}\right\rfloor + 1;$$

$$b_k := \exp\left(iun^{-\frac{1}{2}-\delta_3+\delta_2}\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\xi_y N_{n-1}(y)\right);$$

$$a_k := \exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\sigma_\xi^2(N_{n-1}(y))^2\right).$$

We have to estimate :

$$n^{\delta_2}\left|\mathbb{E}\left[\prod_{k=0}^{L_n} b_k\middle|(S_p)_p\right](\omega) - \prod_{k=0}^{L_n} a_k(\omega)\right|.$$

Hence it is enough to estimate :

$$n^{\delta_2}\sum_{k=0}^{L_n}\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-1} b_m\right)(b_k-a_k)\left(\prod_{m'=k+1}^{L_n} a_{m'}\right)\middle|(S_p)_p\right](\omega)\right|.$$

- We explain how we can restrict our study to the sum over the k such that $(r+1)^4\leq k\leq L_n-1$. Indeed, the number of k that does not satisfy this is equal to $(r+1)^4+1$. Let $k\in\{0,...,L_n\}$. We have :

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{\ell=\alpha+1}^{\alpha+\theta}\xi_\ell N_{n-1}(\ell)\right)^2 |(S_p)_p\right](\omega) &\leq \sum_{\ell=\alpha+1}^{\alpha+\theta}\sum_{m=\alpha+1}^{\alpha+\theta}|\mathbb{E}[\xi_\ell\xi_m]|\; N_{n-1}(\ell)(\omega)N_{n-1}(m)(\omega)\\ &\leq \theta\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|n^{1+2\delta_2}.\end{aligned}$$

Hence we have :

$$\begin{aligned}
\mathbb{E}[|b_k-1||(S_p)_p](\omega) &\le n^{-\frac{1}{2}-\delta_3+\delta_2}\left(\mathbb{E}\left[\left|\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\xi_y N_{n-1}(y)\right||(S_p)_p\right](\omega)\right)\\
&\le n^{-\frac{1}{2}-\delta_3+\delta_2}n^{\frac{\beta}{2}}\sqrt{\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|n^{\frac{1}{2}+\delta_2}}\\
&\le n^{-\delta_3+2\delta_2+\frac{\beta}{2}}\sqrt{\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|}\\
&\le n^{-\frac{3}{4}\delta_3+\frac{3}{2}\delta_2}\sqrt{\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|}
\end{aligned}$$

since we have $\beta<\frac{\delta_3}{2}-\delta_2$. Moreover we have :

$$\begin{aligned}
|a_k(\omega)-1| &\le \frac{1}{2n^{1+2\delta_3-2\delta_2}}\sigma_\xi^2\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}}(N_{n-1}(y)(\omega))^2\\
&\le \frac{1}{2n^{1+2\delta_3-2\delta_2}}n^\beta\sigma_\xi^2 n^{1+2\delta_2}\\
&\le \frac{1}{2}n^{-2\delta_3+4\delta_2+\beta}\sigma_\xi^2\\
&\le \frac{1}{2}n^{-\frac{3}{2}\delta_3+3\delta_2}\sigma_\xi^2.
\end{aligned}$$

From which, we get :

$$n^{\delta_2}\sum_{k=0}^{(r+1)^4-1}\mathbb{E}[|b_k-a_k||(S_p)_p](\omega)+\mathbb{E}[|b_{L_n}-a_{L_n}||(S_p)_p](\omega)\le c_0\left(n^{-\frac{3}{4}\delta_3+\frac{5}{2}\delta_2}+n^{-\frac{3}{2}\delta_3+4\delta_2}\right),\quad (2)$$

with $c_0:=((r+1)^4+1)\sqrt{\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|}+\frac{1}{2}\sigma_\xi^2$. Let us recall that $\frac{5}{3}\delta_2<\frac{1}{2}\delta_3$.
Hence, it remains to estimate :

$$n^{\delta_2}\sum_{k=(r+1)^4}^{L_n-1}\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-1}b_m\right)(b_k-a_k)\prod_{m'=k+1}^{L_n}a_{m'}|(S_p)_p\right]\right|.\quad (3)$$

- Let us control the following quantity :

$$\tilde{B}_n:=n^{\delta_2}\sum_{k=(r+1)^4}^{L_n-1}\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^4}b_m\right)\prod_{j=1}^{3}\left(\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^j}b_m\right)-1\right)\prod_{m'=k-r}^{k-1}b_{m'}(b_k-a_k)\prod_{m'=k+1}^{L}a_{m'}\left|(S_p)_p\right.\right]\right|.$$

We have :

$$\tilde{B}_n(\omega)\le n^{\delta_2}\sum_{k=(r+1)^4}^{L_n-1}\prod_{j=1}^{3}\left\|\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^j}b_m\right)-1\right\|_{L^\infty(U_n)}||b_k-a_k||_{L^\infty(U_n)}.$$

On U_n, we have :

$$|b_k-1|\le n^{-\frac{1}{2}-\delta_3+\delta_2}n^\beta n^{\frac{1}{2}+\delta_2}\le n^{-\delta_3+2\delta_2+\beta}.$$

Analogously, we get :

$$\left|\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^j}b_m\right)-1\right|\le r(r+1)^j n^{-\delta_3+2\delta_2+\beta}.$$

On the other hand, we have :

$$|a_k - 1| \leq \frac{1}{2} n^{-2\delta_3+4\delta_2+\beta} \sigma_\xi^2.$$

Therefore we have :

$$\begin{aligned}\tilde{B}_n &\leq 4n^{\delta_2} n^{\frac{1}{2}+\delta_1-\beta} r^3 (r+1)^6 \left(1 + \frac{1}{2}\sigma_\xi^2\right) \left(n^{-\delta_3+2\delta_2+\beta}\right)^4 \\ &\leq 4\left(1+\frac{1}{2}\sigma_\xi^2\right) r^3 (r+1)^6 n^{\frac{1}{2}-\frac{5}{2}\delta_3+6\delta_2+\delta_1},\end{aligned}$$

according to the fact that $\beta < \frac{\delta_3}{2} - \delta_2$. The control of the quantity $\tilde{B}_n$ comes from the fact that $\frac{5}{2}\delta_3 > \frac{1}{2} + 6\delta_2 + \delta_1$.

- It remains to estimate :

$$n^{\delta_2} \sum_{k=(r+1)^4+1}^{L_n-1} \sum_{1 \leq j_0 < j_1 \leq j_2 \leq 4} C_{n,k,j_0,j_1,j_2}$$

where C_{n,k,j_0,j_1,j_2} is the following quantity :

$$\left| \mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^4} b_m \right) \left(\prod_{m=k-(r+1)^{j_2}+1}^{k-(r+1)^{j_1}} b_m \right) \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_m \right) (b_k - a_k) \prod_{m'=k+1}^{L_n} a_{m'} \right] \right|,$$

with the convention : $\prod_{m=\alpha}^{\beta} b_m = 1$ if $\beta < \alpha$.

- Let j_0, j_1, j_2 be fixed. Let us control C_{n,k,j_0,j_1,j_2}. We have :

$$C_{n,k,j_0,j_1,j_2} \leq D_{n,k,j_0,j_1,j_2} + E_{n,k,j_0,j_1,j_2},$$

where :

$$D_{n,k,j_0,j_1,j_2} := \left| Cov_{|(S_p)_p} \left(\Delta_{n,k,j_1,j_2}, \Gamma_{n,k,j_0} \right) \prod_{m'=k+1}^{L_n} a_{m'} \right|.$$

and

$$E_{n,k,j_0,j_1,j_2} := \left| \mathbb{E}[\, \Delta_{n,k,j_1,j_2} | \, (S_p)_p] \, \mathbb{E}[\, \Gamma_{n,k,j_0} | \, (S_p)_p] \prod_{m'=k+1}^{L_n} a_{m'} \right|.$$

with $\Delta_{n,k,j_1,j_2} := \prod_{m=0}^{k-(r+1)^4} b_m \prod_{m'=k-(r+1)^{j_2}+1}^{k-(r+1)^{j_1}} b_{m'}$ and $\Gamma_{n,k,j_0} := \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_m\right)(b_k - a_k)$.

- Control of the covariance terms.

  Let j_0, j_1, j_2 be fixed. Let $k = (r+1)^4, ..., L_n - 1$. We have :

$$\begin{aligned}D_{n,k,j_0,j_1,j_2} \leq & \left| \mathbb{E}\left[Cov_{|(S_p)_p} \left(\Delta_{n,k,j_1,j_2}, \prod_{m=k-(r+1)^{j_0}+1}^{k} b_m \right) \prod_{m'=k+1}^{L_n} a_{m'} \right] \right| + \\ & + \left| \mathbb{E}\left[Cov_{|(S_p)_p} \left(\Delta_{n,k,j_1,j_2}, \prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_m \right) \prod_{m'=k}^{L_n} a_{m'} \right] \right|.\end{aligned}$$

  But we have :

$$\prod_{m=\theta_1+1}^{\theta_1+\theta_2} b_m = \exp\left(iun^{-\frac{1}{2}-\delta_3+\delta_2} \sum_{\ell=\alpha_{(\theta_1)}}^{\alpha_{(\theta_1+\theta_2+1)}-1} \xi_\ell N_{n-1}(\ell) \right).$$

  Therefore, according to point 4 of the hypothesis of our theorem, we have :

$$D_{n,k,j_0,j_1,j_2} \leq 2C \left(1 + n^{-\frac{1}{2}-\delta_3+\delta_2} \sum_{\ell \in \mathbb{Z}} N_{n-1}(\ell) \right) \varphi_{p,s}$$

with $p := \lfloor n^\beta \rfloor ((r+1)^{j_1} - (r+1)^{j_0})$ and $s := \lfloor n^\beta \rfloor (r+1)^{j_0} - 1$. Let us notice that we have : $p \geq rs$. Hence we have :

$$\begin{aligned} n^{\delta_2} \sum_{k=(r+1)^4}^{L_n - 1} D_{n,k,j_0,j_1,j_2} &\leq 4C \left(n^{1-\delta_3+\delta_1-\beta+2\delta_2} \right) n^{-6\beta} \sup_{s \geq n^\beta} s^6 \varphi_{rs,s} \\ &\leq 4C \left(n^{1-\delta_3+\delta_1-7\frac{\delta_3}{2}+14\delta_2+2\delta_2} \right) \sup_{s \geq n^\beta} s^6 \varphi_{rs,s} \\ &\leq 4C \left(n^{1-\frac{9}{2}\delta_3+\delta_1+16\delta_2} \right) \sup_{s \geq n^\beta} s^6 \varphi_{rs,s} \\ &\leq 4C \left(n^{1-\frac{9}{8}+\delta_1+(\frac{27}{2}+16)\delta_2} \right) \sup_{s \geq n^\beta} s^6 \varphi_{rs,s}, \end{aligned}$$

since $\beta > \frac{\delta_3}{2} - 2\delta_2$ and $\delta_3 > \frac{1}{4} - 3\delta_2$. We finish the control of these terms by noticing that $\delta_1 + (\frac{27}{2} + 16)\delta_2 < \frac{1}{8}$.

- Control of the term with the product of the expectations.

  Let j_0, j_1, j_2 be fixed. Let $k = (r+1)^4, ..., L_n - 1$. We can notice that E_{n,k,j_0,j_1,j_2} is bounded from away by the following quantity :

$$F_{n,k,j_0} := \left| \mathbb{E} \left[\prod_{m=k-(r+1)^{j_0}+1}^{k} b_m - \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_m \right) a_k \middle| (S_p)_p \right] \right|.$$

  We use Taylor expansions of the exponential function.

  - First we explain that in F_{n,k,j_0}, we can replace

$$\prod_{m=k-(r+1)^{j_0}+1}^{k} b_m = \exp \left(iun^{-\frac{1}{2}-\delta_3+\delta_2} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) \right)$$

    by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + iun^{-\frac{1}{2}-\delta_3+\delta_2} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}} \left(\sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) \right)^2 . \quad (4)$$

    Indeed, we control the error by :

$$\frac{1}{6} n^{-\frac{3}{2}-3\delta_3+3\delta_2} \mathbb{E} \left[\left| \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) \right|^3 |(S_p)_p \right].$$

    Moreover, we have :

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) \right|^4 |(S_p)_p \right] &\leq \sum_{y_1,y_2,y_3,y_4=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} |\mathbb{E}[\xi_{y_1}\xi_{y_2}\xi_{y_3}\xi_{y_4}]| \left(n^{\frac{1}{2}+\delta_2} \right)^4 \\ &\leq c_0' n^{2+4\delta_2} (r+1)^6 n^{2\beta}, \end{aligned}$$

    according to the hypothesis of our theorem. Hence, taking the sum over $k = (r+1)^4, ..., L_n - 1$ and multiplying by n^{δ_2}, this substitution induces a total error bounded by :

$$\frac{1}{12} n^{\delta_2+\frac{1}{2}+\delta_1-\beta} n^{-\frac{3}{2}-3\delta_3+3\delta_2} n^{\frac{3}{2}+3\delta_2} (r+1)^{\frac{9}{2}} n^{\frac{3}{2}\beta}$$

and so by :

$$\frac{1}{12}n^{7\delta_2+\frac{1}{2}+\delta_1-3\delta_3+\frac{1}{2}\beta}(r+1)^{\frac{9}{2}}.$$

But, using the facts that $\beta < \frac{\delta_3}{2} - \delta_2$ and that $\delta_3 > \frac{1}{4} - 3\delta_2$, we have :

$$\begin{aligned} 7\delta_2 + \frac{1}{2} + \delta_1 - 3\delta_3 + \frac{1}{2}\beta &\leq \frac{13}{2}\delta_2 + \frac{1}{2} + \delta_1 - \frac{11}{4}\delta_3 \\ &\leq -\frac{3}{16} + \frac{59}{4}\delta_2 + \delta_1 \\ &\leq -\frac{1}{16}, \end{aligned}$$

since $\delta_1 + (\frac{27}{2} + 16)\delta_2 < \frac{1}{8}$.

- Let us introduce $Y_k := \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha(k)-1} \xi_\ell N_{n-1}(\ell)$ and $Z_k := \sum_{\ell=\alpha_{(k)}}^{\alpha(k+1)-1} \sigma_\xi^2 N_{n-1}(\ell)^2$. We explain that, in F_{n,k,j_0}, we can replace

$$\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_m\right) a_k = \exp\left(\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k\right)$$

by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k + \frac{1}{2}\left(\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k\right)^2. \quad (5)$$

Indeed the modulus of the error between these two quantities is less than :

$$\frac{1}{6}\mathbb{E}\left[\left|\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k\right|^3 |(S_p)_p\right] \leq \frac{4}{3}\mathbb{E}\left[\left|\frac{1}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k\right|^3 + \left|\frac{1}{2n^{1+2\delta_3-2\delta_2}}Z_k\right|^3 |(S_p)_p\right].$$

The first term will be controled as just before. Let us control the second term. We have :

$$\begin{aligned} \left|n^{-1-2\delta_3+2\delta_2}Z_k\right|^3 &\leq n^{-3-6\delta_3+6\delta_2}\left(\sigma_\xi^2\right)^3 n^{3\beta} n^{3+6\delta_2} \\ &\leq n^{-6\delta_3+12\delta_2+3\beta}\left(\sigma_\xi^2\right)^3. \end{aligned}$$

Hence, taking the sum over $k = (r+1)^4, ..., L_n - 1$ and multiplying by n^{δ_2}, we get a quantity bounded by :

$$2n^{\frac{1}{2}+\delta_1-6\delta_3+13\delta_2+2\beta}\left(\sigma_\xi^2\right)^3.$$

But we have :

$$\begin{aligned} \frac{1}{2} + \delta_1 - 6\delta_3 + 13\delta_2 + 2\beta &\leq \frac{1}{2} + \delta_1 - 5\delta_3 + 11\delta_2 \\ &\leq \frac{1}{2} - \frac{5}{4} + \delta_1 + 23\delta_2 < 0. \end{aligned}$$

since $\beta < \delta_3 - 2\delta_2$ and $\delta_3 > \frac{1}{4} - 3\delta_2$.

- Now, we show that in formula (5), we can ommit the term with $(Z_k)^2$. Indeed, we have :

$$\begin{aligned} n^{\delta_2}\sum_{(r+1)^4}^{L_n-1}\left(n^{-1-2\delta_3+2\delta_2}Z_k\right)^2 &\leq 2n^{\delta_2+\frac{1}{2}+\delta_1-\beta-2-4\delta_3+4\delta_2}n^{2\beta}(\sigma_\xi^2)^2 n^{2+4\delta_2} \\ &\leq 2n^{\frac{1}{2}+\delta_1-4\delta_3+9\delta_2+\beta}(\sigma_\xi^2)^2 \\ &\leq 2n^{\frac{1}{2}+\delta_1-\frac{7}{2}\delta_3+8\delta_2}(\sigma_\xi^2)^2 \\ &\leq 2n^{-\frac{1}{12}-\frac{1}{6}\delta_1-\frac{1}{6}\delta_2}(\sigma_\xi^2)^2 \end{aligned}$$

since $\beta < \frac{\delta_3}{2} - \delta_2$ and $3\delta_3 > \frac{1}{2} + 7\delta_2 + \delta_1$.

– Hence, it remains to estimate the following quantity called G_{n,k,j_0} :

$$\left|\mathbb{E}\left[\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}(Y_k+W_k)-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left(Y_k+W_k\right)^2-\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k+\right.\right.$$
$$\left.\left.+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}(Y_k)^2+\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k\right|(S_p)_p\right]\right|$$

with $W_k:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\xi_\ell N_{n-1}(\ell)$. We get :

$$\begin{aligned}G_{n,k,j_0} &= \left|\mathbb{E}\left[-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left(Y_k+W_k\right)^2+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}(Y_k)^2\right|(S_p)_p\right]\right|\\ &= \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left|\mathbb{E}\left[\,(W_k)^2+2W_kY_k-Z_k\right|(S_p)_p\right]\right|.\end{aligned}$$

Let us notice that we have :

$$Z_k:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2]N_{n-1}(\ell)^2+2\sum_{m\le\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)^2\right).$$

– Let us show that, in the last expression of G_{n,k,j_0}, we can replace Z_k by :

$$\tilde{Z}_k:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2]N_{n-1}(\ell)^2+2\sum_{m\le\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)N_{n-1}(m)\right).$$

Indeed, we have :

$$\begin{aligned}\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\mathbb{E}\left[\left|Z_k-\tilde{Z}_k\right|\,\middle|(S_p)_p\right] &\le \frac{1}{n^{1+2\delta_3-2\delta_2}}\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\sum_{m\le\ell-1}|\mathbb{E}[\xi_\ell\xi_m]||N_{n-1}(\ell)|N_{n-1}(m)-N_{n-1}(\ell)|\\ &\le \frac{1}{n^{1+2\delta_3-2\delta_2}}n^\beta\sum_{m\ge1}|\mathbb{E}[\xi_0\xi_m]|n^{\frac{1}{2}+\delta_2}\sqrt{m}n^{\frac{1}{4}+\frac{\gamma}{2}}\\ &\le n^{-\frac{1}{4}-2\delta_3+3\delta_2+\beta+\frac{\gamma}{2}}\sum_{m\ge1}\sqrt{m}|\mathbb{E}[\xi_0\xi_m]|.\end{aligned}$$

Hence, taking the sum over $k=(r+1)^4,...,L_n-1$ and multiplying by n^{δ_2}, we get a quantity bounded by :

$$4n^{\frac{1}{4}+\delta_1-2\delta_3+4\delta_2+\frac{\gamma}{2}}\sum_{m\ge1}\sqrt{m}|\mathbb{E}[\xi_0\xi_m]|.$$

But we have :

$$\frac{1}{4}+\delta_1-2\delta_3+4\delta_2+\frac{\gamma}{2}\le-\frac{1}{4}+\delta_1+10\delta_2+\frac{\gamma}{2}<0$$

since $\delta_3>\frac{1}{4}-3\delta_2$ and $\gamma<\frac{1}{2}-22\max(\delta_1,\delta_2)$.

– Hence we have to estimate :

$$\tilde{G}_{n,k,j_0}=\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left|\mathbb{E}\left[\,(W_k)^2+2W_kY_k-\tilde{Z}_k\right|(S_p)_p\right]\right|.$$

We have :

$$\mathbb{E}\left[\,(W_k)^2\middle|(S_p)_p\right]=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2](N_{n-1}(\ell))^2+2\sum_{m=\alpha_{(k)}}^{\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)N_{n-1}(m)\right).$$

Hence we have :

$$\mathbb{E}\left[\,(W_k)^2+2W_kY_k\middle|(S_p)_p\right]=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2](N_{n-1}(\ell))^2+2\sum_{m=\alpha_{k-(r+1)^{j_0}+1}}^{\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)N_{n-1}(m)\right).$$

We get :

$$\begin{aligned}\tilde{G}_{n,k,j_0} &= \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left|\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\sum_{m\le\alpha_{k-(r+1)^{j_0}+1}-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)N_{n-1}(m)\right|\\ &\le \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}n^\beta\sum_{m\ge\frac{(r+1)n^\beta}{2}}|\mathbb{E}[\xi_0\xi_m]n^{1+2\delta_2}\\ &\le \frac{1}{2}n^{-2\delta_3+4\delta_2+\beta}\sum_{m\ge\frac{(r+1)n^\beta}{2}}|\mathbb{E}[\xi_0\xi_m]|.\end{aligned}$$

Hence, taking the sum over $k=(r+1)^4,...,L_n-1$ of these quantities and multiplying by n^{δ_2}, we get a quantity bounded by :

$$2n^{\frac{1}{2}+\delta_1-2\delta_3+5\delta_2}\sum_{m\ge\frac{(r+1)n^\beta}{2}}|\mathbb{E}[\xi_0\xi_m]|\le 2n^{\delta_1+11\delta_2}\sum_{m\ge\frac{(r+1)n^\beta}{2}}|\mathbb{E}[\xi_0\xi_m]|,$$

since $\delta_3>\frac{1}{4}-3\delta_2$. To conclude it suffices to notice that :

$$n^{\delta_1+11\delta_2}\sum_{m\ge\frac{(r+1)n^\beta}{2}}|\mathbb{E}[\xi_0\xi_m]|=O(n^{-\varepsilon}).$$

3.1.2 Proof of lemma 5

Let us consider $n\ge 2$. According to lemma 6, it suffices to prove that there exists a real number $\delta'>0$ such that we have :

$$\sup_{n\ge1}n^{\delta'}\sup_{\omega\in U_n}n^{\delta_2}\int_{|u|\le1}\exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_{y,z}\mathbb{E}[\xi_y\xi_z](N_{n-1}(y)(\omega))^2\right)e^{-\frac{u^2n^{2\delta_2}}{2}}\,du<+\infty.$$

Let us take $\omega\in U_n$. We have :

$$\exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_{y,z}\mathbb{E}[\xi_y\xi_z](N_{n-1}(y)(\omega))^2\right)=\exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sigma_\xi^2\sum_y(N_{n-1}(y)(\omega))^2\right).$$

Let us define : $p_n:=Card\{y\in\mathbb{Z}\ :\ N_{n-1}(y)\ge\frac{n^{\frac{1}{2}-\delta_4}}{3}\}$. We have :

$$\begin{aligned}n &= \sum_{y=-\lfloor n^{\frac{1}{2}+\delta_1}\rfloor}^{\lfloor n^{\frac{1}{2}+\delta_1}\rfloor}N_{n-1}(y)\\ &\le p_nn^{\frac{1}{2}+\delta_2}+\frac{n^{\frac{1}{2}-\delta_4}}{3}\left(3n^{\frac{1}{2}+\delta_1}-p_n\right)\\ &\le p_n\left(n^{\frac{1}{2}+\delta_2}-\frac{n^{\frac{1}{2}-\delta_4}}{3}\right)+n^{\frac{1}{2}-\delta_4}n^{\frac{1}{2}+\delta_1}\\ &\le p_nn^{\frac{1}{2}+\delta_2}\left(1-\frac{n^{-(\delta_2+\delta_4)}}{3}\right)+n^{1+\delta_1-\delta_4}.\end{aligned}$$

Let us recall that : $\delta_1<\delta_4<\frac{1}{4}-\delta_3-\frac{5}{2}\delta_2$. Hence we have :

$$p_n\ge n^{-\frac{1}{2}-\delta_2}\left(n-n^{1-(\delta_4-\delta_1)}\right)\ge n^{\frac{1}{2}-\delta_2}\left(1-n^{-(\delta_4-\delta_1)}\right)\ge c_0n^{\frac{1}{2}-\delta_2},$$

with $c_0 := 1 - 2^{-(\delta_4-\delta_1)}$.

Hence we have :

$$\sum_{y\in\mathbb{Z}} (N_{n-1}(y)(\omega))^2 \geq p_n \left(\frac{n^{\frac{1}{2}-\delta_4}}{3}\right)^2 \geq \frac{c_0 n^{\frac{3}{2}-\delta_2-2\delta_4}}{9}.$$

$$\begin{aligned}\exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_y \sigma_\xi^2 (N_{n-1}(y)(\omega))^2\right) &\leq \exp\left(-\frac{u^2}{18n^{1+2\delta_3-2\delta_2}}\sigma_\xi^2 c_0 n^{\frac{3}{2}-\delta_2-2\delta_4}\right)\\ &\leq \exp\left(-\frac{u^2}{18}\sigma_\xi^2 c_0 n^{\frac{1}{2}-3\delta_2-2\delta_3-2\delta_4}\right).\end{aligned}$$

Therefore, we have :

$$n^{\delta_2}\int_{|u|\leq 1} \exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_{y,z}\mathbb{E}[\xi_y\xi_z](N_{n-1}(y)(\omega))^2\right) e^{-\frac{u^2 n^{2\delta_2}}{2}}\, du \leq$$

$$\begin{aligned}&\leq n^{\delta_2}\int_{|u|\leq 1}\exp\left(-\frac{u^2}{18}\sigma_\xi^2 c_0 n^{\frac{1}{2}-3\delta_2-2\delta_3-2\delta_4}\right) du\\ &\leq \frac{n^{\delta_2}}{n^{\frac{1}{4}-\delta_4-\frac{3}{2}\delta_2-\delta_3}}\int_{\mathbb{R}}\exp\left(-\frac{v^2}{18}\sigma_\xi^2 c_0\right) dv\\ &\leq n^{-\frac{1}{4}+\delta_4+\frac{5}{2}\delta_2+\delta_3}\int_{\mathbb{R}}\exp\left(-\frac{v^2}{18}\sigma_\xi^2 c_0\right) dv.\end{aligned}$$

with the change of variable $v = un^{\frac{1}{4}-\delta_4-\frac{3}{2}\delta_2-\delta_3}$. This gives the result since $\delta_4+\delta_3+\frac{5}{2}\delta_2 < \frac{1}{4}$ *qed.*

4 About the model of Guillotin-Le Ny

In this section, we prove that the hypothesis $\int_M \frac{1}{\sqrt{f(1-f)}}\, d\nu < +\infty$ of Guillotin-Le Ny in [2] can be replaced by $\int_M \frac{1}{(f(1-f))^p}\, d\nu < +\infty$, for some $p > 0$. In this situation, there is no need to introduce the set U_n; we can take $U_n = A_n$. If we take $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$, all the points except the point 3(b)(ii) comes in the same way without the need of the hypothesis $\int_M \frac{1}{\sqrt{f(1-f)}}\, d\nu < +\infty$.

Guillotin and Le Ny have to estimate the following quantity :

$$\sup_{\omega\in A_n} n^{\frac{1}{2}+\delta_3} I_n^{(1)}(\omega) := \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}}\mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}}\xi_y N_{n-1}(y)}\Big|(S_p)_p\right](\omega) e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt.$$

Let us take $\omega \in A_n$.

We will suppose $\delta_3 > 2\delta_2$ and $\delta_1 < \delta_4 < \frac{1}{4} - \delta_3 - \frac{\delta_2}{2}$.

The idea of Guillotin and Le Ny is to write :

$$\begin{aligned}n^{\frac{1}{2}+\delta_3}\left|I_n^{(1)}\right| &\leq n^{\frac{1}{2}+\delta_3}\int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}}\mathbb{E}\left[\prod_{y\in\mathbb{Z}}|\cos(tN_{n-1}(y)) + i(2f\circ T^y - 1)\sin(tN_{n-1}(y))|\,|(S_p)_p\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt\\ &\leq n^{\frac{1}{2}+\delta_3}\int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}}\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\sqrt{1-4f\circ T^y(1-f\circ T^y)\sin^2(tN_{n-1}(y))}|(S_p)_p\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt\\ &\leq n^{\frac{1}{2}+\delta_3}\int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}}\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\sqrt{1-4f\circ T^y(1-f\circ T^y)\frac{16}{\pi^2}(tN_{n-1}(y))^2}|(S_p)_p\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt\end{aligned}$$

$$\leq \quad n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[\prod_{y\in\mathbb{Z}} e^{-8\pi^2 f\circ T^y(1-f\circ T^y)t^2 N_{n-1}(y)^2} |(S_p)_p\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt$$

since $|tN_{n-1}(y)| \leq n^{-\frac{1}{2}-\delta_3+\delta_2} n^{\frac{1}{2}+\delta_2} = n^{2\delta_2-\delta_3}$. Hence, if n is large enough, then $|tN_{n-1}(y)|$ will be uniformly less than $\frac{\pi}{2}$ and $|\sin(tN_{n-1}(y))| \geq \frac{2}{\pi}|tN_{n-1}(y)|$. We also use the fact that, for positive u, we have : $1-u \leq e^{-u}$. According to the Hölder inequality with $\sum_y \frac{N_{n-1}(y)^2}{\sum_k N_{n-1}(k)^2} = 1$, we have :

$$n^{\frac{1}{2}+\delta_3}\left|I_n^{(1)}\right| \leq n^{\frac{1}{2}+\delta_3} \int_{\mathbb{R}\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{-8\pi^2 f(1-f)t^2\sum_k N_{n-1}(k)^2} |(S_p)_p\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt.$$

Now, we use the fact that, since $\delta_4 > \delta_1$, there exists a constant c such that we have :

$$\forall \omega' \in A_n,\quad \sum_y (N_{n-1}(y))^2(\omega') \geq c n^{\frac{3}{2}-\delta_2-2\delta_4}.$$

This has been proved in the previous section called 'proof of lemma 5'. Hence, under the hypothesis $\int_M \frac{1}{\sqrt{f(1-f)}}\, d\nu < +\infty$ of Guillotin and Le Ny, we have :

$$\begin{aligned}
n^{\frac{1}{2}+\delta_3}\left|I_n^{(1)}(\omega)\right| &\leq n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{-8\pi^2 f(1-f)t^2 n^{\frac{3}{2}-\delta_2-2\delta_4}}\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt \\
&\leq n^{-\frac{1}{4}+\delta_3+\frac{\delta_2}{2}+\delta_4} \int_{\mathbb{R}} \mathbb{E}\left[\frac{1}{\sqrt{f(1-f)}}\right] e^{-8\pi^2 v^2}\, dv
\end{aligned}$$

with the change of variable $v = t\sqrt{f(1-f)n^{\frac{3}{2}-\delta_2-2\delta_4}}$. We adapt this argument to our hypothesis. Now let us replace the hypothesis $\int_M \frac{1}{\sqrt{f(1-f)}}\, d\nu < +\infty$ by $\int_M \frac{1}{[f(1-f)]^p}\, d\nu < +\infty$ for some $p > 0$. Let us take $\delta_1 < \delta_4 < \frac{1}{4} - \delta_3 - \frac{\delta_2}{2} - \frac{\delta_2}{p}$. We use the fact that there exists a constant $c_p > 0$ such that, for any real number $u > 0$, we have : $e^{-u} \leq \frac{c_p}{u^p}$. We have :

$$\begin{aligned}
n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{3}{4}+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}}\}} \mathbb{E}\left[e^{-8\pi^2 f(1-f)t^2 n^{\frac{3}{2}-\delta_2-2\delta_4}}\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt &\leq n^{\frac{1}{2}+\delta_3} n^{-\frac{3}{4}+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}} \\
&\leq n^{-\frac{1}{4}+\delta_3+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}}.
\end{aligned}$$

On the other hand, we have :

$$n^{\frac{1}{2}+\delta_3} \int_{\{n^{-\frac{3}{4}+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}} < |t| < n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{-8\pi^2 f(1-f)t^2 n^{\frac{3}{2}-\delta_2-2\delta_4}}\right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt \leq$$

$$\begin{aligned}
&\leq n^{\frac{1}{2}+\delta_3} n^{-\frac{1}{2}-\delta_3+\delta_2} \int_M e^{-8\pi^2 f(1-f) n^{\frac{2\delta_2}{p}}}\, d\nu \\
&\leq n^{-\delta_2} c_p \left(\frac{1}{8\pi^2}\right)^p \int_M [f(1-f)]^{-p}\, d\nu.
\end{aligned}$$

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