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- Birkhoff metric
- Ergodic processes in  $\mathbb{R}^d$
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Nonobservable ergodic diffusion process  $(X_t)$  with

- values in  $\mathbb{R}^d$ ;
- observations  $(Y_t)$  from  $\mathbb{R}^{\ell}$ ;
- initial distribution  $\mu_0$  (of  $X_0$ ) known with some error.

### he question

Is this error forgotten by the optimal filtering algorithm in the long run?

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What does it mean "the optimal filtering algorithm" ?

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# The observation model the precise definition

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## Markov diffusion process:

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$$dX_t = b(X_t)dt + dW_t, \quad (t \ge 0),$$

### observation:

$$dY_t = h(X_t)dt + dV_t \quad (t \ge 0),$$

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where

■  $(W_t, V_t)$  is  $\mathbb{R}^{d+\ell}$  valued Wiener process; ■  $b : \mathbb{R}^d \to \mathbb{R}^d$ ; ■  $h : \mathbb{R}^d \to \mathbb{R}^{\ell}$ ;

# The observation model the precise definition

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$$\mathbf{P}_t^{\mu_0,Y}(\cdot) = \mathcal{P}_{\mu_0}(X_t \in \cdot \mid \mathcal{F}_t^Y),$$

• with 
$$\mathcal{F}_t^{\mathbf{Y}} = \sigma(\mathbf{Y}_s : \mathbf{0} \le s \le t)$$
,  
• with the initial measure  $\mu_0$ .

The strange conditional probability:

$$\mathsf{P}_t^{
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• with  $\mu_0$  replaced by  $\nu_0$ .

### The question for discussion:

Why  $\mathbf{P}_{t}^{\nu_{0},Y}(\cdot)$  is well defined?

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Why  $\mathbf{P}_{t}^{\nu_{0},Y}(\cdot)$  is well defined?

# The main question, formulation

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# The main question:

## True or false:

$$\lim_{t\to\infty} E_{\mu_0} \| \mathbf{P}_t^{\mu_0, Y}(\cdot) - \mathbf{P}_t^{\nu_0, Y}(\cdot) \|_{\tau_V} = 0?$$

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## Stability of filters

# True or false:

$$\lim_{t o \infty} E_{\mu_0}(\pi_t^{\mu_0,Y}(f) - \pi_t^{
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where

■ the true conditional expectation:

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# D.Blackwell, 1957

### he model

Nonobservable **stationary** ergodic **finite** state Markov chain  $(X_n)$ 

• observations  $Y_n = \Phi(X_n)$ 

Φ is not one-to-one.

### he question

Is the stationary measure of the conditional distribution unique?

### A question for discussion

What is the connection with the subject of the talk?

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# Stability and uniqueness

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# Two related questions

Blackwell: Is a stationary measure unique (only *Q*)?We: Is the filter stable?

### act

**Stability** of filter  $\Rightarrow$  **uniqueness** of stationary measure. (A. Budhiraja, H.J.Kushner).

A.Budhiraja (2008) - link between different properties of the nonlinear filter process:

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- Stability of the filter with respect to initial conditions
- Uniqueness of the invariant measure of the filter
- "Finite memory" property of the filter

# Stability and uniqueness

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# 1971, 1991, H. Kunita: "yes" in diffusion model.

Model:

Signal  $X_t$  — ergodic Markov process valued in a locally compact space.

Observations:

$$dY_t = h(X_t)dt + dW_t$$

Claim:

$$\lim_{t\to\infty} E_{\mu_0}(f(X_t)-\pi_t^{\mu_0,Y}(f))^2$$

does non depend on  $\mu_0$ , the invariant measure of the filtering process is unique.

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# The first time, sometimes the answer is "no"

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# **1974, Kaijser** : a counter-example

X<sub>n</sub> - an ergodic Markov chain with S = {1, 2, 3, 4}
 transition matrix

$$\Lambda = \frac{1}{2} \left( \begin{array}{rrrr} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array} \right)$$

• observation (**noiseless**):  $\mathbf{Y}_n = \mathbf{1}_{X_n=1} + \mathbf{1}_{X_n=3}$ 

Result: there is no uniqueness, no stability

 $\lim_{n\to\infty} E_{\mu_0}(\pi_n^{\mu_0,Y}(x)-\pi_n^{\nu_0,Y}(x))^2 \geq C(\mu_0,\nu_0)>0.$ 

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# 1991, Delyon & Zeitouni :

- consider finite state space ergodic signal or linear case;
- introduce the term "memory length" of filters;
- propose a programme of analysis of exponential stability of filters using Lyapounov exponents.

# **1996, D.Ocone, E.Pardoux** :

- consider Kunita's model.
- **Claim**: The optimal filter is stable:

 $\lim_{n \to \infty} E_{\mu_0}(\pi_n^{\mu_0, Y}(f) - \pi_n^{\nu_0, Y}(f))^2 = 0 \quad \forall f \in C_b, \, \nu_0 \sim \mu_0.$ 

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- introduce the term "memory length" of filters;
- propose a programme of analysis of exponential stability of filters using Lyapounov exponents.

# ■ 1996, D.Ocone, E.Pardoux :

- consider Kunita's model.
- **Claim**: The optimal filter is stable:

$$\lim_{n\to\infty} E_{\mu_0}(\pi_n^{\mu_0,\,Y}(f)-\pi_n^{\nu_0,\,Y}(f))^2=0 \quad \forall f\in C_b,\,\nu_0\sim\mu_0.$$

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(the proof is crucially based on the H. Kunita result)

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## 1997, Atar & Zeitouni

- consider discrete and continuous time, compact valued Markov signal;
- use Birkhoff contraction principle.

### 1998, Atar

considers continuous time, one dimensional non-compact case, with linear observations and sufficiently small noise in observations.

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# A bit later

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# 2004 P.Baxendale, P.Chiganskii, R.Liptser Serious gap in Kunita's proof.

The Kunita's proof was based on the following:

### Frue or false

for an **ergodic** Markov process *X*<sub>t</sub>?

# A bit later

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The Kunita's proof was based on the following:

### True or false

$$\bigcap_{n\geq 1} \mathcal{F}_{[0,\infty)}^{Y} \bigvee \mathcal{F}_{[n,\infty)}^{X} = \mathcal{F}_{[0,\infty)}^{Y}$$

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for an **ergodic** Markov process X<sub>t</sub>?

# Nothing is clear

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# Counterexample for the proof

an **ergodic** Markov process  $X_t$  with

- state space  $S = \{1, 2, 3, 4\};$
- transition intensity matrix:

$$\Lambda = \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

**noiseless** observation:  $\mathbf{Y}_n = \mathbf{1}_{X_n=1} + \mathbf{1}_{X_n=3}$ .

**Result: the answer is "False".** Also, filter is unstable, the invariant measure of the filtering process is not unique.

# Nothing is clear

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- Budhiraja & Ocone (1999), Oudjane & Rubenthaler (2005) — discrete time, small observation noise.
- Stannat (2005) Continuous time case, a gradient type drift and linear observation part under additional assumptions.
- Liptser, Chigansky (2005,2006,2007) continuous time compact case, exponential stability via Lyapounov exponents.
- van Handel (2008)- Kunita's proof is revised

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## Reminder

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### The model

- Markov diffusion process:  $dX_t = b(X_t)dt + dW_t$ ,
- observation:  $dY_t = h(X_t)dt + dV_t$ .

### The question

### True or false:

$$\lim_{t \to \infty} E_{\mu_0} \| \mathbf{P}_t^{\mu_0, Y}(\cdot) - \mathbf{P}_t^{\nu_0, Y}(\cdot) \|_{\tau_V} = 0?$$

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# Assumptions, I

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## (A0) *b* is locally bounded;

(A1<sub>ρ</sub>) : the signal is **recurrent**(Khasminskii-Veretennikov conditions):

$$= 0: \qquad \limsup_{|x|\to\infty} \left\langle b(x), \frac{x}{|x|} \right\rangle \le -r, \ r > 0$$

0

$$= 1 : \lim_{|x| \to \infty} \langle b(x), x \rangle = -\infty.$$

### ixamples

 $(p = 0): b(x) = -sign(x), b(x) = -x; \dots$ 

 $(p = 1): b(x) = -\frac{\arctan(x)}{\sqrt{1+|x|}};$ 

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Coupling and separation The main inequality Sketch of the proof part 2 (A0) *b* is locally bounded; (A1<sub>p</sub>) : the signal is **recurrent** (Khasminskii-Veretennikov conditions): p = 0:  $\limsup_{|x| \to \infty} \left\langle b(x), \frac{x}{|x|} \right\rangle \le -r, r > 0$ or

# p = 1: $\lim_{|x| \to \infty} \langle b(x), x \rangle = -\infty.$

### xamples

$$(p = 0) : b(x) = -sign(x), b(x) = -x; \dots$$

$$(p = 1) : b(x) = -\frac{\arctan(x)}{\sqrt{1 + |x|}};$$

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### Examples

$$(p = 0): b(x) = -sign(x), b(x) = -x; \dots$$

$$(p=1): b(x) = -rac{\arctan(x)}{\sqrt{1+|x|}}; \dots$$

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# (A2) The function *h* is smooth enough: $h \in C^2$ , & $\|\nabla h\|_{C^1} < \infty$ .

## the first state of a set of a state base of the set of the

(A3) : Initial data is absolutely continuous.



(A4) Initial moments are finite:

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#### Sketch of the proof

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$$h \in C^2$$
, &  $\|\nabla h\|_{C^1} < \infty$ .

## (A3) : Initial data is absolutely continuous.

$$\left\|\frac{d\mu_0}{d\nu_0}\right\|_{L_{\infty}(\nu_0)}<\infty.$$

(A4) Initial moments are finite:

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## (A3) : Initial data is absolutely continuous.

$$\left\|\frac{d\mu_0}{d\nu_0}\right\|_{L_{\infty}(\nu_0)} < \infty.$$

(A4) Initial moments are finite:

 $\int e^{c|x|}\mu_0(dx) < \infty.$ 

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## Stability with bounds - main result

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### Theorem

Under Assumptions (A0) – (A4) the following bounds hold:

$$E_{\mu_0} \| \boldsymbol{P}_t^{\mu_0,Y}(\cdot) - \boldsymbol{P}_t^{\nu_0,Y}(\cdot) \|_{\tau v} \leq \left\{ \begin{array}{ll} C_m t^{-m}, & p=1, \quad \forall m>0, \\ C \exp(-ct), & p=0. \end{array} \right.$$

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# Conditional distribution, a particular case

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### A pair

Let the pair (X, Y) be the solution of:

$$\begin{cases} dX_s = b(Y, X_s) ds + dW_s, \\ dY_s = dB_s, \end{cases}$$

## with independent (W, B)

### ts first component

and let  $X_s^{\psi}$  be s.t. (with **deterministic**  $\psi$ ) :

$${ extsf{d}} X^\psi_{m{s}} = { extsf{b}}(\psi, X^\psi_{m{s}}) \, { extsf{d}} {m{s}} + { extsf{d}} W_{m{s}}$$

Then the **conditional** law  $\mathcal{L}(X \mid Y)$  is just the **law** of  $X^{\psi}$  with a **substitution**  $\psi = Y$ .

# Conditional distribution, a particular case

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and let  $X_s^{\psi}$  be s.t. (with **deterministic**  $\psi$ ) :

$$dX^\psi_{m{s}} = m{b}(\psi, X^\psi_{m{s}})\, dm{s} + dW_{m{s}}$$

Then the **conditional** law  $\mathcal{L}(X \mid Y)$  is just the **law** of  $X^{\psi}$  with a **substitution**  $\psi = Y$ .

# Conditional distribution, a particular case

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# Cauchy problem

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### **Diffusion process**

$$dX_t^{\psi} = b(\psi(t), X_t^{\psi})dt + dW_t, \quad (t \ge 0),$$

Then  $E_x f(X_t^{\psi}) \exp[\int_0^t c(s, X_s^{\psi}) ds] = u^{\psi}(0, x)$  is the solution of:

### Cauchy problem

$$u_s + \Delta u/2 + b(\psi(s), x)\nabla u + c(s, x)u = 0, \quad u(t, x) = f(x)$$

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Continuity properties of the solution w.r.t  $\psi$  are known.

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### A bounded domain

Let

$$D_0 := \{ \sup_{0 \le s \le 1} |X_s^{\psi}| < R+1 \}.$$

Then  $E_x\left(1(D_0)f(X_1^{\psi})\right)\exp[\int_0^1 c(s,X_s^{\psi})ds] = u^{\psi}(0,x) -$ solution of

### he first boundary problem

 $u_s + \frac{1}{2}\Delta u + b(\psi, x)\nabla u + c(s, x)u = 0,$  $u^{\psi}(1, x) = f(x); \quad u^{\psi}(s, x) = 0, \ 0 < s < 1, \ |x| = R + 1.$ 

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# Harnack's inequality

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Coupling and separation The main inequality Sketch of the proof, part 2 Let *u* ≥ 0 be a solution of the first boundary problem with ■ **uniformly bounded** coefficients,

■ final condition in a cylinder

$$\{(t, x): 0 < t < 1; |x| < \mathbf{R+1}, \}.$$

Variant of Harnack's inequality: Krylov, Safonov, 1980

$$\sup_{x|,|z|\leq \mathbf{R}} \ \frac{u(0,x)}{u(0,z)} \leq C_{\scriptscriptstyle R}$$

where  $C_R$  depends only on R and on the upper bounds of the coefficients.

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# Birkhoff metric

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### Definition

The Birkhoff distance between positive measures:

$$p(\mu, 
u) = \left\{egin{array}{ll} {\sf ln} \sup({d\mu}/{d
u}) + {\sf ln} \sup({d
u}/{d\mu}), & {\sf if finite,} \ +\infty, & {\sf otherwise.} \end{array}
ight.$$

**Remark.** It is a pseudo-distance, measuring the difference between directions.

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## Comparison of total variation distance and Birkhoff distance

(Christophe Leuriden, private communication)

For normalized measures  $\mu$  and  $\nu$ :

$$\|\mu - \nu\|_{\mathrm{TV}} \le \rho(\mu, \nu)$$

The converse statement does not hold.

### xample

$$egin{array}{rcl} q_{\mu}(x) &=& {f 1}\,(x\in [-1/2,1/2]) \ q_{
u}(x) &=& rac{1}{2}\cdot {f 1}\,(|x|\in [arepsilon,1/2]) + C\cdot {f 1}\,(x\in [-arepsilon,arepsilon]) \end{array}$$

Then  $\|\mu - \nu\|_{\tau \nu} = 1 - 2\varepsilon$ ,  $\rho(\mu, \nu) = \ln(1 + \frac{2}{\varepsilon})$ 

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# Birkhoff contraction for nonnegative kernels: Let $\mathbf{Q} : \mathcal{M}(R^d) \to \mathcal{M}(R^d)$ s.t.: $\mu \mathbf{Q}(dy) = \int_{R^d} Q(x, dy) \mu(dx)$ .

### Contraction

$$\rho(\mu \mathbf{Q}, \nu \mathbf{Q}) \leq \frac{C^2 - 1}{C^2 + 1} \rho(\mu, \nu), \text{ with}$$

(Krasnosel'skii, Lifshits, Sobolev)

 $C = \sup_{x,z,y} \frac{q(x,y)}{q(z,y)}, \ Q(x,dy) = q(x,y)dy$ 

( Le Gland, Oudjane

$$C = \sup_{x,z,A} \frac{Q(x,A)}{Q(z,A)}.$$

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# Ergodic processes in $\mathbb{R}^d$ , properties

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## Hitting time estimates (A. Veretennikov, 1987):

For  $\hat{\tau} = \inf(t \ge 0 : |X_t| \le R)$ 

$$\left\{\begin{array}{ll}E_x\hat{\tau}^k \leq C_m(1+|x|^m) & (\forall \ m>2k; p=1),\\E_x\exp(\alpha\hat{\tau}) \leq C\exp(c|x|) & (p=0).\end{array}\right.$$

### Corollary

Let 
$$\Lambda(X)_R := \sum_{k=0}^{n-1} \mathbf{1}(|X_k| \le R).$$

Then ( $\forall 0 < \varepsilon < 1$  and for R large enough)

$$E_{\mu_0} \mathbb{1}(\Lambda(X)_R < \varepsilon n) \leq \begin{cases} C_m n^{-m}, & (p=1), \\ C \exp(-cn), & (p=0) \end{cases}$$

# Ergodic processes in $\mathbb{R}^d$ , properties

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- Reformulation of the problem, the Bayes approach

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# The Bayes formula, part 1

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## The classical Bayes formula, I

$$\boldsymbol{P}(\boldsymbol{X}_t \in \cdot \mid \mathcal{F}_t^{\boldsymbol{Y}}) = \frac{\widehat{\boldsymbol{E}}(\boldsymbol{1}(\boldsymbol{X}_t \in \cdot)\boldsymbol{L}_t(\overline{\boldsymbol{X}}, \overline{\boldsymbol{Y}}) \mid \mathcal{F}_t^{\boldsymbol{Y}})}{\widehat{\boldsymbol{E}}\boldsymbol{L}_t(\overline{\boldsymbol{X}}, \overline{\boldsymbol{Y}}) \mid \mathcal{F}_t^{\boldsymbol{Y}})},$$

with

$$L_t(\overline{X},\overline{Y}) = \frac{dP}{d\widehat{P}}(\overline{X},\overline{Y})$$

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# The Bayes formula, II

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### Changing of measure

Using **Girsanov's transformations and integration by parts** we change the measure:

$$\frac{dP}{d\hat{P}} = \exp[\sum_{k=1}^{[t]} h^*(X_k)(Y_k - Y_{k-1}) + h^*(X_t)(Y_t - Y_{[t]})]$$

$$+\frac{1}{2}\int_0^t c(X_s,Y)\,ds],$$

### with

 $c(s, x, Y) = \|(Y_s - Y_{[s]})^* \nabla h(x)\|^2 - 2(Y_s - Y_{[s]})^* \Delta h(x) - \|h\|^2(x).$ 

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$$L_n = \prod_{k=1}^n \exp[h^*(X_k)(Y_k - Y_{k-1}) + \frac{1}{2} \int_{k-1}^k c(X_s, Y) \, ds],$$

The transformed process (X, Y) (w.r.t  $\hat{P}$ ) is nice:

 $\begin{cases} dX_s = (b(X_s) - (Y_s - Y_{[s]})^* \nabla h(X_s)) ds + dW_s, \\ dY_s = dB_s, \end{cases}$ 

with independent W and B.

- We are in the situation "Conditional distribution, particular case"
- Now we can choose the continuous (w.r.t *Y*) version of the conditional measure.

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■ Hence, we can use the first boundary problem.

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### Exact filtering algorithm via a nonlinear integral operator

$$\bar{\mu}_t(\cdot;\mu_0) = \mathbf{P}_t^{\mu_0,Y}(\cdot) =: \mu_0 \mathbf{Q}_t^Y(\cdot)$$

Its explicit form  $\bar{\mu}(A; \mu_0) = c_t^{\mu_0} \int_{R^d} Q_t(x_0, A) d\mu_0(x_0)$ , with

 $Q_t(x_0, A) = \widehat{E}_{x_0}(\mathbf{1}(X_t \in A)L_t(\overline{X}, \overline{Y}) \mid \mathcal{F}_t^Y)$ 

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Q(x<sub>0</sub>, A) can be found from the Cauchy problem.
 c<sup>µ</sup><sub>t</sub> - normalizing coefficient, gives the nonlinearity, (the denominator in the Bayes formula).

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## Main question - reformulation

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### Strange conditional probability via the same operator:

$$\mathbf{P}_t^{\nu_0,Y}(\cdot) =: \nu_0 \mathbf{Q}_t^Y(\cdot) = \frac{\mathbf{c}_t^{\nu_0}}{\int\limits_{\mathbf{R}^d} Q_t(x_0, \mathbf{A}) \, d\nu_0(x_0).$$

### Main question - reformulation

Frue or false:

$$\lim_{t\to\infty} E_{\mu_0} \|\mu_0 \mathbf{Q}_t^{\boldsymbol{Y}}(\cdot) - \nu_0 \mathbf{Q}_t^{\boldsymbol{Y}}(\cdot)\|_{\tau V} = 0?$$

## Main question - reformulation

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$$\mathbf{P}_{t}^{\nu_{0},Y}(\cdot) =: \nu_{0}\mathbf{Q}_{t}^{Y}(\cdot) = \frac{c_{t}^{\nu_{0}}}{\int_{R^{d}}} Q_{t}(x_{0},A) \, d\nu_{0}(x_{0}).$$

### Main question - reformulation

### True or false:

$$\lim_{t\to\infty} E_{\mu_0} \|\mu_0 \mathbf{Q}_t^Y(\cdot) - \nu_0 \mathbf{Q}_t^Y(\cdot)\|_{\tau_V} = 0?$$

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## Coupling, doubling the space

Consider **independent** couples (X, Y) and  $(\tilde{X}, \tilde{Y})$  with initial laws  $\mathcal{L}(X_0) = \mu_0$ ,  $\mathcal{L}(\tilde{X}_0) = \nu_0$ .

### Doubling the operators,

New operators on the space of measures on  $R^{2d}$  $\bar{\mu}_l(A \times B; (\mu_0, \nu_0)) = c_l^{\mu_0} c_l^{\nu_0} \int Q_l(x_0, \tilde{x}_0; A \times B) d\mu_0(x_0) d\nu_0(\tilde{x}_0).$ 

with

 $\begin{aligned} Q_t(x_0, \tilde{x}_0; A \times B) &= \widehat{E}_{x_0, \tilde{x}_0}(1(X_t \in A, \tilde{X}_t \in B) \\ &\times L_t(X, Y) L_t(\tilde{X}, \tilde{Y}) \mid \mathcal{F}_t^{Y, \tilde{Y}}) \Big|_{\tilde{Y} = Y} \end{aligned}$ 

## Using coupling method, I A. Veretennikov, Lecture Notes, 2004

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 $egin{aligned} \mathcal{Q}_t(x_0, ilde{x}_0;\, A imes B) &= \widehat{\mathcal{E}}_{x_0, ilde{x}_0}(\mathbf{1}(X_t\in A, ilde{X}_t\in B)\ & imes L_t(X,Y)L_t( ilde{X}, ilde{Y}) \mid \mathcal{F}_t^{Y, ilde{Y}}) igg|_{ ilde{Y}=Y} \end{aligned}$ 

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## Remark. The substitutions are well defined.

### Comparison of measures

The following properties hold:

- $\blacksquare \bar{\mu}_t(\boldsymbol{A}; \mu_0) = \bar{\mu}_t(\boldsymbol{A} \times \boldsymbol{R}^d; (\mu_0, \nu_0))$
- $\blacksquare \bar{\mu}_t(\boldsymbol{A};\nu_0) = \bar{\mu}_t(\boldsymbol{A}\times\boldsymbol{R}^d;(\nu_0,\mu_0))$

### Comparison of distances

 $\|\bar{\mu}_t(\cdot;\mu_0) - \bar{\mu}_t(\cdot;\nu_0)\|_{\tau_V} \le \|\bar{\mu}_t(\cdot;(\mu_0,\nu_0)) - \bar{\mu}_t(\cdot;(\nu_0,\mu_0))\|_{\tau_V}$ 

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### Comparison of measures

### The following properties hold:

$$\bar{\mu}_t(A; \mu_0) = \bar{\mu}_t(A \times R^d; (\mu_0, \nu_0))$$

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### Comparison of distances

$$\|\bar{\mu}_t(\cdot;\mu_0) - \bar{\mu}_t(\cdot;\nu_0)\|_{\tau_V} \le \|\bar{\mu}_t(\cdot;(\mu_0,\nu_0)) - \bar{\mu}_t(\cdot;(\nu_0,\mu_0))\|_{\tau_V}$$

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### Partition of unity

For fixed *R*, *n*, and any non-random vector  $\delta \in \Delta = \{0, 1\}^{n+1}$  define

$$\mathbf{1}_{\delta}(X, ilde{X}):=\prod_{i=0}^{n-1}\left(\mathbf{1}\left(D_{i}
ight)
ight)^{\delta_{i}} imes\left(\mathbf{1}-\mathbf{1}\left(D_{i}
ight)
ight)^{\mathbf{1}-\delta_{i}},$$

#### vhere

$$D_i := \left\{ \max\left(|X_i|, |\tilde{X}_i|\right) \le R; 
ight.$$
$$\left( \sup_{i \le s \le i+1} |X_s|, \sup_{i \le s \le i+1} |\tilde{X}_s| 
ight) < R+1 
ight\}$$

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### Multiplicative decomposition

$$\mathbf{1}_{\delta}(X, ilde{X}):=\prod_{i=0}^{n-1}\mathbf{1}_{\delta_i}(D_i)$$

with

$$\mathbf{1}_{\delta_i}(D_i) = \mathbf{1}(\delta_i = 1)\mathbf{1}(D_i) + \mathbf{1}(\delta_i = 0)(1 - \mathbf{1}(D_i)).$$

### Partition of unity

$$\mathbf{1} = \sum_{\delta \in \Delta} \mathbf{1}_{\delta}(X, \tilde{X})$$

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# Separation of pairs

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Coupling and separation The main inequality Sketch of the proof, part 2 Denote by  $\#1(\delta)$  the total number of **ones** in  $\delta$  and by

$$\#1(X)_R := \sum_{k=0}^{n-1} \mathbf{1}(|X_k| \le R, \sup_{k \le s \le k+1} |X_s| < R+1,)$$

The following inequalities hold:

### Separation of pairs,

$$\sum_{i=1(\delta)<\varepsilon n} \mathbf{1}_{\delta}(X,\tilde{X}) \leq \mathbf{1}(\#\mathbf{1}(X)_R < \frac{1+\varepsilon}{2}n) + \mathbf{1}(\#\mathbf{1}(\tilde{X})_R < \frac{1+\varepsilon}{2}n)$$

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## Then ( $\forall \varepsilon > \frac{1}{2}$ and for *R* large enough)

### Separation of pairs,II

$$E_{\mu_0}\mathbf{1}((\#\mathbf{1}(X)_R < \varepsilon n) \leq \begin{cases} C_m n^{-m}, & (p=1), \\ C \exp(-cn), & (p=0) \end{cases}$$

The proof is based on the hitting time estimates, exponential Chebyshev's inequality and the fact that

 $q = \sup_{x:|x| \le R} P_x(\sup_{0 \le s \le +1} |X_s| \ge R+1) < 1/2.$ 

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## Our goal is to prove the following inequality:

### The main inequality

$$E_{\mu_0}\|\bar{\mu}_t(\cdot;\mu_0)-\bar{\mu}_t(\cdot;\nu_0)\|_{\tau V} \leq C \sum_{\delta \in \Delta} \kappa_{R}^{\#1(\delta)} E_{\mu_0,\nu_0} e_{[t]}^{\mathbf{Y};\delta;\mu_0,\nu_0},$$

with  $u(s, x, \tilde{x})$  - the solution of the first boundary problem.  $s_{aaa}$ 

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$$E_{\mu_0}\|\bar{\mu}_t(\cdot;\mu_0)-\bar{\mu}_t(\cdot;\nu_0)\|_{_{TV}} \leq C\sum_{\delta\in\Delta}\kappa_{_R}^{\#1(\delta)}E_{\mu_0,\nu_0}e_{[t]}^{Y;\delta;\mu_0,\nu_0},$$

with  $u(s, x, \tilde{x})$  - the solution of the first boundary problem.

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### First boundary problem

$$\begin{split} u_{s} + \frac{1}{2}u_{xx} + \frac{1}{2}u_{\tilde{x}\tilde{x}} \\ + (b(x) - (\psi_{s} - \psi_{0})^{*}\nabla h(x))u_{x} + (b(\tilde{x}) - (\psi_{s} - \psi_{0})^{*}\nabla h(\tilde{x}))u_{\tilde{x}} \\ + \frac{1}{2}c(x, \tilde{x}, \psi)u = 0, \\ u(1, x, \tilde{x}) = (\mathbf{1}(x \in A, \tilde{x} \in B) \\ \times \exp[h^{*}(x)(\psi_{1} - \psi_{0}) + h^{*}(\tilde{x})(\psi_{1} - \psi_{0})] \\ u(s, x, \tilde{x}) = 0, \ \forall \ 0 < s < 1, \ \max(|x|, |\tilde{x}| = R + 1), \end{split}$$

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with a replacement  $\psi = Y$ .

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Sketch of the proof part 2 The term  $e_t^{Y;\delta;\mu_0,\nu_0}$  in the main inequality is defined by:

### Probability separator, definition

$$e_t^{Y;\delta;\mu_0,\nu_0} := E_{\mu_0,\nu_0}(\mathsf{1}_{\delta}(X,\tilde{X}) \mid Y,\tilde{Y})\Big|_{\tilde{Y}=Y}.$$

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Remark. This term will be the normalizing coefficient.

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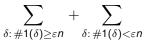
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Sketch of the proo part 2 We split the sum in the main inequality ( $\forall \varepsilon > 0$ ):



and we estimate both terms:

 $\sum_{\delta: \, \#1(\delta) \ge \varepsilon n} \kappa_{R}^{\#1(\delta)} E_{\mu_{0}} e_{n}^{Y;\delta;\mu_{0},\nu_{0}} \le \kappa_{R}^{\varepsilon n}$ 

 $\leq \sum_{\delta:\,\#\,1(\delta)<\varepsilon n} E_{\mu_0}\left(E_{\mu_0,\nu_0}(\mathbb{1}_{\delta}(X,\tilde{X})\mid Y,\tilde{Y})\Big|_{\tilde{Y}=Y}\right).$ 

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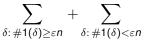
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$$\sum_{\delta: \#1(\delta) < \varepsilon n} \kappa_{R}^{\#1(\delta)} E_{\mu_{0}} \left( E_{\mu_{0},\nu_{0}}(1_{\delta}(X,\tilde{X}) \mid Y,\tilde{Y}) \Big|_{\tilde{Y}=Y} \right)$$

$$\leq \sum_{\delta: \, \# \, \mathbf{1}(\delta) < \varepsilon n} E_{\mu_0} \left( E_{\mu_0, \nu_0}(\mathbf{1}_{\delta}(X, \tilde{X}) \mid Y, \tilde{Y}) \Big|_{\tilde{Y} = Y} \right).$$

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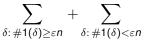
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 $\sum_{\delta: \#1(\delta) \geq \varepsilon n} \kappa_{\scriptscriptstyle R}^{\#1(\delta)} E_{\mu_0} \boldsymbol{e}_{\scriptscriptstyle D}^{\boldsymbol{Y};\delta;\mu_0,\nu_0} \leq \kappa_{\scriptscriptstyle R}^{\varepsilon n}$ 

 $\sum_{R} = \kappa_{R}^{\#1(\delta)} E_{\mu_{0}} \left( E_{\mu_{0},\nu_{0}}(1_{\delta}(X,\tilde{X}) \mid Y,\tilde{Y}) \Big|_{\tilde{v}} \right)$ 

 $\leq \sum_{\mu_0} \left| E_{\mu_0,\nu_0}(\mathbf{1}_{\delta}(X,\tilde{X}) \mid Y,\tilde{Y}) \right|_{\mathfrak{s}_{\mu_0}} \right).$ 

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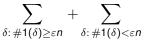
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 $\sum_{\delta: \#1(\delta) \geq \varepsilon n} \kappa_{R}^{\#1(\delta)} \mathcal{E}_{\mu_{0}} \boldsymbol{e}_{n}^{\boldsymbol{Y};\delta;\mu_{0},\nu_{0}} \leq \kappa_{R}^{\varepsilon n}$ 

$$\sum_{\delta: \#1(\delta) < \varepsilon n} \kappa_{R}^{\#1(\delta)} E_{\mu_{0}} \left( E_{\mu_{0},\nu_{0}}(1_{\delta}(X,\tilde{X}) \mid Y,\tilde{Y}) \Big|_{\tilde{Y}=Y} \right)$$

$$\leq \sum_{\delta: \, \# \, \mathbf{1}(\delta) < \varepsilon n} \, \mathcal{E}_{\mu_0} \left( \mathcal{E}_{\mu_0, \nu_0}(\mathbf{1}_{\delta}(X, \tilde{X}) \mid Y, \, \tilde{Y}) \Big|_{\tilde{Y}=Y} \right).$$

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+

$$egin{aligned} & E_{\mu_0}\left(E_{\mu_0,
u_0}\left(\sum_{\delta:\,\#1(\delta)$$

(because X does not depend on  $\tilde{Y}$ , nor  $\tilde{X}$  depends on Y).

■ the inequality "separation of pairs" has been used.

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### We estimate the first term

$$E_{\mu_0}\left(E_{\mu_0}\left(1(\#1(X)_R < \frac{1+\varepsilon}{2}n) \mid Y\right)\right)$$
$$= E_{\mu_0}\left(1(\#1(X)_R < \frac{1+\varepsilon}{2}n)\right).$$

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we can use the "Separation of pairs, II".

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### We estimate the first term

$$\begin{split} E_{\mu_0}\left(E_{\mu_0}\left(1(\#1(X)_R < \frac{1+\varepsilon}{2}n) \mid Y\right)\right) \\ &= E_{\mu_0}\left(1(\#1(X)_R < \frac{1+\varepsilon}{2}n)\right). \end{split}$$

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we can use the "Separation of pairs, II".

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# Next, we estimate the other term, using the **absolute continuity of the initial measures**:

$$E_{\mu_0}\left(E_{\nu_0}\left(1(\#1(\tilde{X})_R < \frac{1+\varepsilon}{2}n) \mid \tilde{Y}\right)\Big|_{\tilde{Y}=Y}\right)$$
  
$$\leq C_2 E_{\nu_0}\left(E_{\nu_0}\left(1(\#1(\tilde{X})_R < \frac{1+\varepsilon}{2}n) \mid \tilde{Y}\right)\Big|_{\tilde{Y}=Y}\right)$$
  
$$= C_2 E_{\nu_0}\left(1(\#1(\tilde{X})_R < \frac{1+\varepsilon}{2}n)\right),$$

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Again, the Separation of pairs, II.

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Sketch of the proof part 2 Next, we estimate the other term, using the **absolute continuity of the initial measures**:

$$egin{aligned} & E_{\mu_0}\left(E_{
u_0}\left(1(\#1( ilde{X})_R < rac{1+arepsilon}{2}n) \mid ilde{Y}
ight)\Big|_{ ilde{Y}=Y}
ight) \ & \leq C_2\,E_{
u_0}\left(E_{
u_0}\left(1(\#1( ilde{X})_R < rac{1+arepsilon}{2}n) \mid ilde{Y}
ight)\Big|_{ ilde{Y}=Y}
ight) \ & = C_2\,E_{
u_0}\left(1(\#1( ilde{X})_R < rac{1+arepsilon}{2}n)
ight), \end{aligned}$$

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$$egin{aligned} & E_{\mu_0}\left(E_{
u_0}\left(1(\#1( ilde{X})_R < rac{1+arepsilon}{2}n) \mid ilde{Y}
ight)\Big|_{ ilde{Y}=Y}
ight) \ & \leq C_2\,E_{
u_0}\left(E_{
u_0}\left(1(\#1( ilde{X})_R < rac{1+arepsilon}{2}n) \mid ilde{Y}
ight)\Big|_{ ilde{Y}=Y}
ight) \ & = C_2\,E_{
u_0}\left(1(\#1( ilde{X})_R < rac{1+arepsilon}{2}n)
ight), \end{aligned}$$

Again, the Separation of pairs, II.

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### How can we prove the main inequality?

Define new **linear** operators on the space of non-normalized measures on  $\mathbb{R}^{2d}$ 

$$\mu_t(A \times B; (\mu_0, \nu_0)) = \int_{B^{2d}} Q_t(x_0, \tilde{x}_0; A \times B) \, d\mu_0(x_0) \, d\nu_0(\tilde{x}_0).$$

with the same kernel  $Q_t$ :

$$\begin{aligned} Q_t(x_0, \tilde{x}_0; A \times B) &= \widehat{E}_{x_0, \tilde{x}_0}(\mathbf{1}(X_t \in A, \tilde{X}_t \in B) \\ & \times L_t(X, Y) L_t(\tilde{X}, \tilde{Y}) \mid \mathcal{F}_t^{Y, \tilde{Y}}) \Big|_{\widetilde{Y} = Y} \end{aligned}$$

We have

$$\bar{\mu}_t(A \times B; (\mu_0, \nu_0)) = \mathbf{C}_t^{\mu_0} \mathbf{C}_t^{\nu_0} \mu_t(A \times B; (\mu_0, \nu_0))$$

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$$\mu_t(\boldsymbol{A}\times\boldsymbol{B};(\mu_0,\nu_0))=\int_{\boldsymbol{R}^{2d}}Q_t(\boldsymbol{x}_0,\tilde{\boldsymbol{x}}_0;\,\boldsymbol{A}\times\boldsymbol{B})\,d\mu_0(\boldsymbol{x}_0)\,d\nu_0(\tilde{\boldsymbol{x}}_0).$$

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We have

$$\bar{\mu}_t(A \times B; (\mu_0, \nu_0)) = \mathbf{C}_t^{\mu_0} \mathbf{C}_t^{\nu_0} \mu_t(A \times B; (\mu_0, \nu_0))$$

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$$\mu_t(\boldsymbol{A}\times\boldsymbol{B};(\mu_0,\nu_0))=\int_{\boldsymbol{R}^{2d}}Q_t(\boldsymbol{x}_0,\tilde{\boldsymbol{x}}_0;\,\boldsymbol{A}\times\boldsymbol{B})\,d\mu_0(\boldsymbol{x}_0)\,d\nu_0(\tilde{\boldsymbol{x}}_0).$$

with the same kernel  $Q_t$ :

$$egin{aligned} egin{aligned} \mathcal{Q}_t(x_0, ilde{x}_0;oldsymbol{A} imes \mathcal{B}) &= \widehat{\mathcal{E}}_{x_0, ilde{x}_0}(oldsymbol{1}(X_t\in \mathcal{A}, ilde{X}_t\in \mathcal{B}) \ & imes L_t(X,Y)L_t( ilde{X}, ilde{Y}) \mid \mathcal{F}_t^{Y, ilde{Y}}) \Bigert_{ ilde{Y}=Y} \end{aligned}$$

Ne have

$$\bar{\mu}_t(A \times B; (\mu_0, \nu_0)) = \boldsymbol{c}_t^{\mu_0} \boldsymbol{c}_t^{\nu_0} \mu_t(A \times B; (\mu_0, \nu_0))$$

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Sketch of the proof, part 2 How can we prove the main inequality? Define new **linear** operators on the space of non-normalized measures on  $\mathbb{R}^{2d}$ 

$$\mu_t(A \times B; (\mu_0, \nu_0)) = \int_{\mathcal{H}^{2d}} Q_t(x_0, \tilde{x}_0; A \times B) \, d\mu_0(x_0) \, d\nu_0(\tilde{x}_0).$$

with the same kernel  $Q_t$ :

$$egin{aligned} egin{aligned} \mathcal{Q}_t(x_0, ilde{x}_0;oldsymbol{A} imes \mathcal{B}) &= \widehat{\mathcal{E}}_{x_0, ilde{x}_0}(oldsymbol{1}(X_t\in\mathcal{A}, ilde{X}_t\in\mathcal{B}) \ & imes L_t(X,Y)L_t( ilde{X}, ilde{Y}) \mid \mathcal{F}_t^{Y, ilde{Y}}) \Bigert_{ ilde{Y}=Y} \end{aligned}$$

### We have

$$\bar{\mu}_t(\boldsymbol{A}\times\boldsymbol{B};(\mu_0,\nu_0)) = \boldsymbol{c}_t^{\mu_0}\boldsymbol{c}_t^{\nu_0}\mu_t(\boldsymbol{A}\times\boldsymbol{B};(\mu_0,\nu_0))$$

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Sketch of the proof, part 2 Using the partition of unit we obtain the following decomposition:

$$\mu_t(oldsymbol{A} imes oldsymbol{B};(\mu_0,
u_0)) = \sum_{\delta\in\Delta} \mu_t^\delta(oldsymbol{A} imes oldsymbol{B};(\mu_0,
u_0))$$

### with

 $\mu_t^{\delta}(\boldsymbol{A}\times\boldsymbol{B};(\mu_0,\nu_0))=\int_{\boldsymbol{H}^{2d}}\boldsymbol{Q}_t^{\delta}(\boldsymbol{x}_0,\tilde{\boldsymbol{x}}_0;\boldsymbol{A}\times\boldsymbol{B})\,d\mu_0(\boldsymbol{x}_0)\,d\nu_0(\tilde{\boldsymbol{x}}_0).$ 

and with the kernel  $Q_t^{\delta}$ :

 $egin{aligned} Q_t^\delta(x_0, ilde x_0;\, A imes B) &= \widehat{E}_{x_0, ilde x_0}(\mathbf{1}(X_t\in A, ilde X_t\in B)\mathbf{1}_\delta(oldsymbol{X}, ilde X)\ & imes L_t(X,Y)L_t( ilde X, ilde Y)\mid \mathcal{F}_t^{Y, ilde Y})\Bigert_{Y=Y} \end{aligned}$ 

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Using the partition of unit we obtain the following decomposition:

$$\mu_t(oldsymbol{A} imes oldsymbol{B};(\mu_0,
u_0)) = \sum_{\delta\in\Delta}\mu_t^\delta(oldsymbol{A} imes oldsymbol{B};(\mu_0,
u_0))$$

### with

$$\mu_t^{\delta}(\boldsymbol{A}\times\boldsymbol{B};(\mu_0,\nu_0)) = \int_{\boldsymbol{R}^{2d}} \boldsymbol{Q}_t^{\delta}(\boldsymbol{x}_0,\tilde{\boldsymbol{x}}_0;\boldsymbol{A}\times\boldsymbol{B}) \, \boldsymbol{d}\mu_0(\boldsymbol{x}_0) \, \boldsymbol{d}\nu_0(\tilde{\boldsymbol{x}}_0).$$

and with the kernel  $Q_t^{\delta}$ :

$$\begin{aligned} Q_t^\delta(x_0, \tilde{x}_0; A \times B) &= \widehat{E}_{x_0, \tilde{x}_0}(\mathbf{1}(X_t \in A, \tilde{X}_t \in B) \mathbf{1}_{\delta}(X, \tilde{X}) \\ & \times L_t(X, Y) L_t(\tilde{X}, \tilde{Y}) \mid \mathcal{F}_t^{Y, \tilde{Y}}) \Big|_{\widetilde{Y} = Y} \end{aligned}$$

# Probability separator, again

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# We see that the normalizing coefficient is exactly the $e_t^{Y;\delta;\mu_0,\nu_0}$ :

### Probability separator, II

$$egin{aligned} e_t^{Y;\delta;\mu_0,
u_0} &:= E_{\mu_0,
u_0}(1_{\delta}(X, ilde{X}) \mid Y, ilde{Y}) \Big|_{ ilde{Y}=Y} \ &= c_t^{\mu_0} c_t^{
u_0} \mu_t^{\delta}(R^{2d};(\mu_0,
u_0)) \end{aligned}$$

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# Recursion, I

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Sketch of the proof, part 2 Using the Markov property of  $X_t$  we find the recursion (with  $Z_t = (X_t, \widetilde{X}_t)$ ):

$$\mu_n^{\delta_n}(dz_n) = \int_{R^2d} Q^{\delta_n}(z_{n-1}, dz_n) d\mu_{n-1}^{\delta_{n-1}}(z_{n-1}),$$

### with

$$Q^{\delta_n}(z_{n-1}, D) = E_{z_{n-1}} \mathbf{1}(Z_n \in D) \mathbf{1}_{\delta_n}(D_n) \exp[\int_{n-1}^n c(s, Z_s, Y) \, ds]$$

### with

$$D_n := \left( |Z_{n-1}| \leq R, \sup_{n-1 \leq s \leq n} |Z_s| < R+1 
ight),$$

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# Comparison of distance,2

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Sketch of the proof, part 2 We can estimate the total variation norm:

$$\begin{split} \|\bar{\mu}_{t}(\cdot;(\mu_{0},\nu_{0})) - \bar{\mu}_{t}(\cdot;(\nu_{0},\mu_{0}))\|_{TV} \\ &\leq c_{t}^{\mu_{0}}c_{t}^{\nu_{0}}\sum_{\delta\in\Delta}\|\mu_{t}^{\delta}(\mu_{0},\nu_{0}) - \mu_{t}^{\delta}(\nu_{0},\mu_{0})\|_{TV} \\ &= \sum_{\delta\in\Delta}e_{n}^{Y;\delta;\mu_{0},\nu_{0}}\|\hat{\mu}_{t}^{\delta}(\mu_{0},\nu_{0}) - \hat{\mu}_{t}^{\delta}(\nu_{0},\mu_{0})\|_{TV} \end{split}$$

### with normalization

1

$$\hat{\mu}_t^{\delta}(
u_0,\mu_0) = rac{\mu_t^{\delta}(\mu_0,
u_0)}{\mu_t^{\delta}({\mathcal R}^{{ extsf{2d}}};\mu_0,
u_0)}$$

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$$\begin{split} \|\bar{\mu}_{t}(\cdot;(\mu_{0},\nu_{0})) - \bar{\mu}_{t}(\cdot;(\nu_{0},\mu_{0}))\|_{TV} \\ &\leq \boldsymbol{c}_{t}^{\mu_{0}}\boldsymbol{c}_{t}^{\nu_{0}}\sum_{\delta\in\Delta} \|\mu_{t}^{\delta}(\mu_{0},\nu_{0}) - \mu_{t}^{\delta}(\nu_{0},\mu_{0})\|_{TV} \\ &= \sum_{\delta\in\Delta} \boldsymbol{e}_{n}^{Y;\delta;\mu_{0},\nu_{0}} \|\hat{\mu}_{t}^{\delta}(\mu_{0},\nu_{0}) - \hat{\mu}_{t}^{\delta}(\nu_{0},\mu_{0})\|_{TV} \end{split}$$

with normalization

$$\hat{\mu}_t^{\delta}(
u_0,\mu_0) = rac{\mu_t^{\delta}(\mu_0,
u_0)}{\mu_t^{\delta}(\mathcal{R}^{\mathsf{2d}};\mu_0,
u_0)}$$

# Using the Birkhoff metric, 1

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### Using the properties of the Birkhoff metric we see that

### Birkhoff metric, 1st property.

$$\|\hat{\mu}_t^{\delta}(\mu_0,\nu_0) - \hat{\mu}_t^{\delta}(\nu_0,\mu_0)\|_{\tau \nu} \le \rho(\hat{\mu}_t^{\delta}(\mu_0,\nu_0);\hat{\mu}_t^{\delta}(\nu_0,\mu_0)).$$

# Using the Birkhoff metric, 2

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### and that

### Birkhoff metric, 2nd property.

$$egin{aligned} &
ho(\hat{\mu}_n^{\delta}(\mu_0,
u_0);\hat{\mu}_n^{\delta}(
u_0,\mu_0))\ &\equiv
ho\left(\mu_n^{\delta}(\mu_0,
u_0);\mu_n^{\delta}(
u_0,\mu_0)
ight)\ &\leq\kappa_R^{\delta_n}
ho\left(\mu_{n-1}^{\delta}(\mu_0,
u_0);\mu_{n-1}^{\delta}(
u_0,\mu_0)
ight)\ &\leq C\kappa_R^k, \end{aligned}$$

with

$$k = \#1(\delta)$$

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which gives the desired inequality.