The Continuous Time Nonzero-sum Dynkin Game and Applications.

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1. Introduction

- $(\Omega, \mathcal{F}, (F_t)_{t\le T}, P)$ is a filtered probability space with $(F_t)_{t\le T}$ complete and right continuous ; T is the horizon of the problem.

- Two players a_1 and a_2 act on a system up to the time when one of them decides to stop controlling, at a stopping time τ_1 (resp. τ_2) for a_1 (resp. a_2).

- The reward for a_1 (resp. a_2) is given by

$$J_1(\tau_1, \tau_2) \overset{\triangle}{=} E\Big\{X^1_{\tau_1} 1_{\{\tau_1 \le \tau_2\}} + Y^1_{\tau_2} 1_{\{\tau_2 < \tau_1\}}\Big\}$$

(resp.

$$J_2(\tau_1, \tau_2) \overset{\triangle}{=} E\Big\{X^2_{\tau_2} 1_{\{\tau_2 < \tau_1\}} + Y^2_{\tau_1} 1_{\{\tau_1 \le \tau_2\}}\Big\}).$$

Definition: A pair (τ_1^*, τ_2^*) of F_t-stopping times is called a Nash equilibrium point for the NZSDG if it satisfies: $\forall \tau_1, \tau_2 \in \mathcal{T}_0$,

$$J_1(\tau_1, \tau_2^*) \leq J_1(\tau_1^*, \tau_2^*)$$

and

$$J_2(\tau_1^*, \tau_2) \leq J_2(\tau_1^*, \tau_2^*).$$

Particular case: $J_1 + J_2 = 0$ corresponds to the zero-sum Dunkin game and a NEP for the game is just a saddle point for the ZSDG. It satisfies: $\forall\ \tau_1, \tau_2$,

$$J_1(\tau_1^*, \tau_2) \leq J_1(\tau_1^*, \tau_2^*) \leq J_1(\tau_1, \tau_2^*).$$

2. Known Results

A. PDE approach (Bens.-Fried., 77)

Assume:

- $\zeta := (\zeta_t)_{t \leq T}$ is a solution of a standard differential equation whose generator is $\mathcal{A}$

- $X_t^i = \varphi^i(t, \zeta_t)$ and $Y^i = \psi^i(t, \zeta_t)$ where ψ^i and φ^i deterministic functions

- **[H1]**: $X^i \leq Y^i$

- **[H2]**: Y^i supermartingales.

Theorem: There exist two deterministic continuous bounded functions $u^1(t,x)$ and $u^2(t,x)$ solution of the following system:

$$\begin{cases} u^i(T,x) = \psi^i(T,x); \\ \\ u^i \geq \varphi^i; \\ \\ if\ u^j(t,x) = \varphi^j(t,x) \text{ for } j \neq i \\ \quad \text{and some } (t,x), \text{ then } u^i(t,x) = \psi^i(t,x); \\ \\ if\ \Sigma^i = \{(t,x), u^j(t,x) > \varphi^j(t,x) \text{ for } j \neq i\}, \\ \quad \text{then } \mathcal{A}u^i(t,x) \geq 0 \text{ for } (t,x) \in \Sigma^i; \\ \\ (u^i - \varphi^i).\mathcal{A}u^i(t,x) = 0 \text{ in } \Sigma^i \end{cases} \tag{1}$$

and the following pair of stopping times,

$$\widehat{\tau}_i = \inf\{s \geq 0, u^i(s,\zeta_s) = \varphi^i(s,\zeta_s)\} \wedge T; i = 1,2$$

is a NEP for the NZSDG.

B. The probabilistic approach (Etourn., 86)

Theorem: The processes are general and satisfy [H1]-[H2]. Then the NZSDG has a NEP.

The proof uses the notion of Snell envelope of processes which is the following:

Let U be an RCLL adapted stochastic process. The Snell envelope of U, denoted by R(U), is the smallest $\mathcal{F}_t$-supermartingale which dominates U, $i.e$, if $\bar{W}$ is another RCLL supermartingale such that $\bar{W}_t \geq U_t$ for all $0 \leq t \leq T$, then $\bar{W}_t \geq W_t$ for any $0 \leq t \leq T$.

It satisfies the following properties:

(i) For any $\mathbf{F}$-stopping time θ we have:

$$W_\theta = \operatorname*{esssup}_{\tau \geq \theta} E[U_\tau | \mathcal{F}_\theta] \ P - a.s. (W_T = U_T);$$

(ii) Assume that U has only positive jumps. Then the stopping time

$$\tau^* \stackrel{\triangle}{=} \inf\{s \geq 0, W_s = U_s\} \wedge T$$

is optimal, $i.e.$,

$$E[W_0] = E[W_{\tau^*}] = E[U_{\tau^*}] = \sup_{\tau \in \mathcal{T}_0} E[U_\tau].$$

As a by-product we have $W_{\tau^*} = U_{\tau^*}$ and the process W is a martingale on the time interval $[0, \tau^*]$.

The main idea of Etourneau's proof is:

Let $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) be the set of RCLL adapted processes V^1 (resp. V^2) such that $X^1 \leq V^1 \leq Y^1$ (resp. $X^2 \leq V^2 \leq Y^2$).

For $(i,j) = (1,2)$ and for $V^j \in \mathcal{E}_j$

$$D_j = \inf\{s \geq 0, V_s^j = X_s^j\} \wedge T$$

and

$$f_i(V^j) = R(X^i 1_{[[0,D_j[[} + Y^i 1_{[[D_j,T]]}).$$

Then [H2] makes that:

(i) f_i is a map from $\mathcal{E}_j$ to $\mathcal{E}_i$,

(ii) $f_i(V^j) = R(V^j) 1_{[[0,D_j[[} + Y^i 1_{[[D_j,T]]}$

Therefore the mappings $f_1 o f_2$ and $f_2 o f_1$ have fixed points W^1 and W^2 which provide a NEP for the NZSDG whoce expression is:

$$\tau^1 = \text{inf}\{s \geq 0, W_s^1 = X_s^1\} \wedge T$$

and

$$\tau^2 = \text{inf}\{s \geq 0, W_s^2 = X_s^2\} \wedge T.$$

<u>3. The main result</u>: without [H2].

Theorem: Assume:

- [H1] i.e. $X^1 \leq Y^1$ and $X^2 \leq Y^2$

- for any stopping time τ,

$$P[\{X^1_\tau < Y^1_\tau\} - \{X^2_\tau < Y^2_\tau\}] = 0$$

(assumption which is satisfied if $X^2 < Y^2$).

- $X^1_T = Y^1_T$ (technical and can be removed).

Then the NZSDG has a NEP (τ^*_1, τ^*_2).

Sketch of the proof:

Let $\tau_1 = T$ and $\tau_2 = T$. For $n = 1, \cdots$, assume τ_{2n-1} and τ_{2n} defined, we then define τ_{2n+1} and τ_{2n+2} as: Let

$$W_t^{2n+1} = \operatorname*{esssup}_{\tau \geq t} E[X_\tau^1 1_{\{\tau < \tau_{2n}\}} + Y_{\tau_{2n}}^1 1_{\{\tau \geq \tau_{2n}} | F_t]$$

$$\tilde{\tau}_{2n+1} = \inf\{t \geq 0 : W_t^{2n+1} = X_t^1\} \wedge \tau_{2n}$$

and

$$\tau_{2n+1} \begin{cases} \tilde{\tau}_{2n+1}, & \text{if} \quad \tilde{\tau}_{2n+1} < \tau_{2n}; \\ \tau_{2n-1}, & \text{if} \quad \tilde{\tau}_{2n+1} = \tau_{2n}. \end{cases}$$

Next, let

$$W_t^{2n+2} = \text{esssup}_{\tau \geq t} \\ E[X_\tau^2 1_{\{\tau < \tau_{2n+1}\}} + Y_{\tau_{2n+1}}^2 1_{\{\tau \geq \tau_{2n+1}} | F_t],$$

$$\tilde{\tau}_{2n+2} = \inf\{t \geq 0 : W_t^{2n+2} = X_t^2\} \wedge \tau_{2n+1}$$

and

$$\tau_{2n+2} = \begin{cases} \tilde{\tau}_{2n+2}, & \text{if } \tilde{\tau}_{2n+2} < \tau_{2n+1}; \\ \tau_{2n}, & \text{if } \tilde{\tau}_{2n+2} = \tau_{2n+1}. \end{cases}$$

The sequences $(\tau_{2n})_{n\geq 0}$ and $(\tau_{2n+1})_{n\geq 0}$ are decreasing and converge respectively to τ_1^* and τ_2^* respectively and (τ_1^*, τ_2^*) is a NEP for the NZSDG.

Step 1: for any stopping time τ,

$$J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n})$$

and

$$J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2}).$$

By definition of W^{2n+1},

- $W^{2n+1}_{\tau_{2n}} = Y^1_{\tau_{2n}}$
- $W^{2n+1}_t \geq X^1_t$ for any $t \in [0, \tau_{2n}]$
- W^{2n+1} is a supermartingale over $[0, \tau_{2n}]$.

Then, for any τ,

$$J_1(\tau, \tau_{2n}) = E\Big\{X^1_\tau 1_{\{\tau \le \tau_{2n}\}} + Y^1_{\tau_{2n}} 1_{\{\tau_{2n} < \tau\}}\Big\}$$
$$\le E\Big\{W^{2n+1}_\tau 1_{\{\tau \le \tau_{2n}\}} + W^{2n+1}_{\tau_{2n}} 1_{\{\tau_{2n} < \tau\}}\Big\}$$
$$= E\{W^{2n+1}_{\tau_{2n}\wedge\tau}\} \le W^{2n+1}_0.$$

But

$$J_1(\tau_{2n+1}, \tau_{2n}) =$$
$$E\Big\{X^1_{\tau_{2n+1}} 1_{\{\tau_{2n+1} \le \tau_{2n}\}} + Y^1_{\tau_{2n}} 1_{\{\tau_{2n} < \tau_{2n+1}\}}\Big\}$$
$$= E\Big\{X^1_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n}\}} + Y^1_{\tau_{2n}} 1_{\{\tau_{2n} \le \tau_{2n+1}\}}\Big\}.$$

Then

$$J_1(\tau_{2n+1}, \tau_{2n}) =$$
$$E\Big\{X^1_{\tilde\tau_{2n+1}} 1_{\{\tilde\tau_{2n+1} < \tau_{2n}\}} + W^{2n+1}_{\tau_{2n}} 1_{\{\tilde\tau_{2n+1} = \tau_{2n}\}}\Big\}$$
$$= E\{W^{2n+1}_{\tilde\tau_{2n+1}}\} = W^{2n+1}_0.$$

Therefore

$$J_2(\tau_{2n+1}, \tau) \le J_2(\tau_{2n+1}, \tau_{2n+2}).$$

In the same way we have the other inequality.

Step 2: Take the limit to obtain the desired result.

4. Application in game options

Assume we have an American game contingent claim whose payoff is:

$$\Gamma(\tau,\sigma) = L_\sigma 1_{[\sigma\le\tau,\sigma<T]} + U_\tau 1_{[\tau<\sigma]} + \xi 1_{[\tau=\sigma=T]}.$$

· $L \le U$ and the difference $U - L$ is the compensation that a_1 pays to a_2 for the decision to terminate the contract before maturity date T.

In a complete market the value of the GCC is given by :

$$V_0 = \sup_{\sigma\ge 0}\inf_{\tau\ge 0} E^*[\Gamma(\tau,\sigma)]$$

$$= \inf_{\tau\ge 0}\sup_{\sigma\ge 0} E^*[\Gamma(\tau,\sigma)].$$

In incomplete markets another point of view is related to utility maximization of the agents (Kuhn, 03).

Let $\varphi_1, \varphi_2 : I\!R \to I\!R$ be two utility functions of the seller, respectively, the buyer of the GCC. The seller a_1 (resp. the buyer a_2) chooses a stopping time τ (resp. σ) in order to maximize

$$J_1(\tau, \sigma) := E[\varphi_1(-\Gamma(\tau, \sigma))]$$

(resp.

$$J_2(\tau, \sigma) := E[\varphi_2(\Gamma(\tau, \sigma))]).$$

Therefore if the NZSDG associated with J_1 and J_2 has a NEP point (σ^*, τ^*), i.e.,

$$J_1(\tau^*, \sigma^*) \geq J_1(\tau, \sigma^*) \text{ and } J_2(\tau^*, \sigma^*) \geq J_2(\tau^*, \sigma)$$

then $-\varphi_1^{-1}(J_1(\tau^*, \sigma^*))$ (resp. $\varphi_2^{-1}(J_2(\tau^*, \sigma^*)))$ is a seller (resp. buyer) price of the GCC.

Theorem: Assume that:

(i) The utility functions φ_1 and φ_2 are non-decreasing;

(ii) L , U are continuous and $L_t \leq U_t$ and $L_T \leq \xi \leq U_T$, P-a.s.;

Then the nonzero-sum Dynkin game associated with the GCC has a Nash equilibrium point (τ^*, σ^*).

Thanks a lot for your attention.