Perfect Foresight Dynamics
An Interface of Differential Games and Game Dynamics

Josef Hofbauer
University of Vienna

A model introduced by *Matsui and Matsuyama*, for a population of rational players who maximize their discounted future payoff,
its associated differential game, and equilibrium selection.


[MM] *Akihiko Matsui and Kiminori Matsuyama:*
An approach to equilibrium selection.
JET 65 (1995), 415-434.

[HS1] *J. Hofbauer and Gerhard Sorger:* Perfect foresight and equilibrium selection in symmetric potential games. JET 85 , 1-23 (1999).

[HS2] *J. Hofbauer and G. Sorger:* A differential game approach to evolutionary equilibrium selection. IGTR 4 (2002) 17-31.

[O] *D. Oyama:* p-Dominance and Equilibrium Selection under Perfect Foresight Dynamics. JET 107 (2002), 289-310.


Differential games
in economics and
management science
Engelbert Dockner · Steffen Jørgensen
Ngo Van Long · Gerhard Sorger

[OTH] *D. Oyama, S. Takahashi and J. Hofbauer:* Monotone methods for equilibrium selection under perfect foresight dynamics. Theoretical Economics 3 (June 2008), 155 - 192.

[KT] *Fuhito Kojima, S. Takahashi:* p-Dominance and Perfect Foresight Dynamic. JEBO 67 (Sept 2008) 689-701.

[T] **Satoru Takahashi:** Perfect Foresight Dynamics in Games with Linear Incentives and Time Symmetry. IJGT 37 (April 2008) 15-38.

[R] *M. Rapp:* Anticipating Cycles. 2008

(Finite) Strategic Games

$N-$person game: payoff function $U : S_1 \times S_2 \times \ldots S_N \rightarrow \mathbb{R}^N$

$$U^i(s_1, s_2, \ldots, s_N)$$

$N-$linear extension to mixed strategies:

$$U : \Delta_1 \times \Delta_2 \times \cdots \times \Delta_N \rightarrow \mathbb{R}^N$$

2-Person games (bimatrix games):

$$U^1(x, y) = x \cdot Ay, \qquad U^2(x, y) = x \cdot By$$

Symmetric 2 person games: $B = A^T$

$$U(x, y) = x \cdot Ay$$

Perfect foresight paths[MM]

N populations of players: $x^i(t) \in \Delta(S_i)$ for $t \geq 0$
random matching, players have perfect foresight and maximize expected discounted payoff

$$V_s^i(t) = \int_0^\infty \int_t^{t+z} e^{-\theta(\tau-t)} U^i(s, x_{-i}(\tau)) d\tau e^{-z} dz$$

$$= \int_t^\infty e^{-(1+\theta)(\tau-t)} U^i(s, x_{-i}(\tau))\, d\tau$$

and switch only to an optimal strategy

$$s \in M^i(t) = \text{argmax}\{V_s^i(t) : s \in S_i\}.$$

$$\begin{array}{rcll} \dot{x}_s^i(t) & = & -x_s^i(t) & \text{if } s \notin M^i(t), \\ \dot{x}_s^i(t) & \in & [-x_s^i(t), 1 - x_s^i(t)] & \text{if } s \in M^i(t) \end{array}$$

$$\begin{aligned} \dot{x}^i_s(t) &= -x^i_s(t) && \text{if } s \notin M^i(t), \\ \dot{x}^i_s(t) &\in [-x^i_s(t), 1 - x^i_s(t)] && \text{if } s \in M^i(t) \end{aligned}$$

$x : [0, \infty) \mapsto \Delta(S_1) \times \cdots \times \Delta(S_N)$ Lipschitz

perfect foresight equilibrium path

for the game U and discount rate θ

The discounted game[HS2]

$$U^i_\theta(x(\cdot)) = \int_0^\infty e^{-\theta s} U^i(x(s))ds \qquad (1)$$

θ–discounted expected payoff for player population i along $x(\cdot)$

initial point $x_0 \in \Delta(S_1) \times \cdots \times \Delta(S_N)$
admissible paths: $X = X_1 \times \cdots \times X_N$

$$X_i = \{x^i : [0, \infty) \to \Delta(S_i),\ \text{Lipschitz}, x^i(0) = x^i_0,$$

$$\dot{x}^i(t) + x^i(t) \in \Delta(S_i) \text{ for a.a. } t \geq 0\}.$$

$\bar{x}(\cdot) = (\bar{x}^i(\cdot))_{i=1}^N \in X$ is an *θ–equilibrium path*
(or open loop Nash equilibrium) if for all $x^i(\cdot) \in X_i$ and all i,

$$U^i_\theta(\bar{x}^i(\cdot); \bar{x}_{-i}(\cdot)) \geq U^i_\theta(x^i(\cdot); \bar{x}_{-i}(\cdot)) \qquad (2)$$

Basic Results [HS2, O]

1. Existence of equilibrium paths

For each initial value $x_0 \in \Delta$ there exists an open loop Nash equilibrium.

Proof: X_i is convex and compact in the topology of uniform convergence on compact intervals. (Ascoli–Arcela)

$U_\theta : X \to \mathbb{R}^N$ continuous, linear in $x^i(\cdot)$. For $x \in X$ and i,

$$\beta^i(x_{-i}) := \operatorname*{argmax}_{x^i(\cdot)\in X^i} \quad U^i_\theta(x^i(\cdot); x_{-i}(\cdot)) \tag{3}$$

is a compact and convex subset of X^i and depends upper semi-continuously on x_{-i}

Schauder–Kakutani fixed point theorem

2. **Each open loop Nash equilibrium path is a perfect foresight equilibrium path and conversely.**

PFE path: bounded Lipschitz solutions $x(t), t \geq 0$ of system

$$\begin{aligned} \dot{x}^i_s &\in m^i_s(V) - x^i_s \\ \dot{V}^i_s &= (\theta + 1)V^i_s - U^i(s, x_{-i}), \end{aligned} \tag{4}$$

$m^i(V) =$ set of optimal mixed strategies for player i

$\bar{x}(\cdot)$ OLNE: $\forall i$, given $\bar{x}_{-i}(\cdot)$, $\bar{x}^i(\cdot)$ is an optimal trajectory of

$$\dot{x}^i = u^i - x^i, \quad u^i \in \Delta(S_i) \tag{5}$$

$$\int_0^\infty e^{-\theta t} U^i(x^i(t), \bar{x}_{-i}(t))dt \to \max \tag{6}$$

Pontrjagin maximum principle,
limiting transversality condition
converse: N-linearity

Example: symmetric 2x2 games

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad (a, b > 0)$$

$$\dot{p}(t) = H(v(t)) - p(t)$$

$$\dot{v}(t) = (1 + \theta)v(t) + \widehat{p} - p(t)$$

$$p = x_2 = 1 - x_1,\ v = (V_2 - V_1)/(a + b), \widehat{p} = a/(a + b)$$

Only 5 bounded solutions: 3 equilibria + stable manifolds

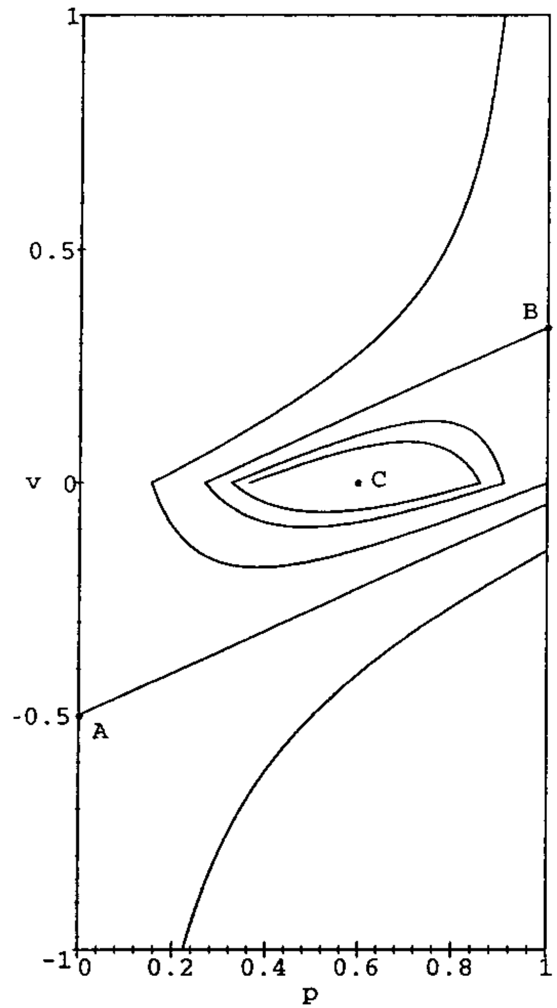

FIG. 1. Phase portrait of (26) for $a = 0.6$, $b = 0.4$, and $\theta = 0.2$.

Stability

The NE a^* is **absorbing** if $\exists$ neighborhood U of a^*: all PF-paths starting in U converge to a^*.

The NE a^* is **d-absorbing** if the only PF-path starting in a^* is the constant one $x(t) = a^*$.

Conj: absorbing $\Leftrightarrow$ d-absorbing

The NE a^* is **globally accessible** if $\forall$ initial state $\exists$ PF-path that converges to a^*.

Neither concept implies the other. (cf. Multiplicity of PF-paths.)

If an absorbing state is also globally accessible then it is the unique absorbing state.

$\longrightarrow$ A method for selecting among equilibria.

<u>2×2 Case</u> [MM] The risk-dominant equilibrium is uniquely absorbing and globally accessible for small discount rate $\theta > 0$.

$\frac{1}{2}$–dominance

$\widehat{s} = (\widehat{s}_i) \in S_1 \times S_2 \times \cdots \times S_N$ is $\frac{1}{2}$–*dominant*: $\widehat{s} = BR(x)$ for all x with $x^i_{\widehat{s}_i} \geq \frac{1}{2}$ for all i.

U has **linear incentives** if $U^i(s, x_{-i}) - U^i(s', x_{-i})$ is linear in x_{-i} $\forall s, s' \in S_i, \forall i$ (e.g. 2 person games)

Theorem. In a game with linear incentives a $\frac{1}{2}$-dominant strategy $\widehat{s}$ is globally accessible for small $\theta > 0$ and absorbing for all $\theta > 0$.

prime example: risk dominance in symmetric 2×2 game

Ellison (2000): for KMR, Young

Proof: 1) The straight path

$$x(t) = x_0 e^{-t} + (1 - e^{-t})\widehat{s} \tag{7}$$

is a PFE path for each x_0.

2) For x_0 close to $\widehat{s}$ the straight path (7) is the only PFE path.

Potential games

$U^i(x) = U(x)$ (or linearly equivalent games)

Let $U(\bar{x}) > U(x)$ for all $x \neq \bar{x}$, i.e. the potential function $U(x)$ has a unique global maximum at $\bar{x}$. Then $\bar{x}$ is globally accessible for small $\theta > 0$ and absorbing for all $\theta > 0$.

Proof: optimal solutions of $U_\theta(x(\cdot)) \to \max$ for $x(\cdot) \in X$ are OLNE = PFE paths; technical, see [HS1, HS2]

The global potential maximizer $\bar{x}$ is selected also by the *global games* method of Carlsson & van Damme (Ui, 2000), but not generally by KMR, HarsanyiSelten risk dominance, etc

Consequences

2×2 coordination games [MM 95]:

$$\begin{pmatrix} a_1, b_1 & 0, 0 \\ 0, 0 & a_2, b_2 \end{pmatrix} \qquad (a_i, b_i > 0)$$

risk dominant equilibrium E: $a_1 b_1 > a_2 b_2$

1) For small $\theta > 0$: E globally accessible

2) for all $\theta > 0$: E absorbing.

Open problem:

$n \times n$ coordination game with payoffs $a_i, b_i > 0$

Is the NE with the highest Nash product $a_i b_i$ selected?

N–person symmetric binary games

Ex: N-person stag hunt   Carlsson–van Damme (1993)

Youngse Kim (GEB 1996): compares 5 methods of equilibrium selection, 4 different criteria

a_i (b_i): payoff for A (B), if i of N players use A
$d(p) = U(B,p) - U(A,p)$ incentive function, $p =$ freq. of B

B **is selected over** A **iff:**   ($n = 2$ risk–dominance)

$\int_0^1 d(p)dp > 0 \Leftrightarrow \sum b_i > \sum a_i$	:	MM, CvD	Pot, logit
$d(p) > 0$ for $\frac{1}{2} \leq p \leq 1$	:	KMR	Güth-K-89, S-95, H–BR
$\int_0^1 p(1-p)d(p)dp > 0$	:	FY-90	H–RE
nonlinear condition in a_i, b_i	:	HS-88	

More than 2 strategies per player

Few results ($\frac{1}{2}$ dominance), many open problems

[T] **Every two-player game has at most one d-absorbing strict Nash equilibrium. This is then globally accessible.** (also true for N person games with linear incentives)

A binary 4 person game with two strict Nash equilibria, both are d-absorbing.

$0, 0, 0, 0$	$0, -1, 0, 0$
$-1, 0, 0, 0$	$-1, -1, 0, 0$

$0, 0, 0, -1$	$0, -1, 0, 1$
$-1, 0, 0, 1$	$-1, -1, 0, 1$

$0, 0, -1, 0$	$0, -1, 1, 0$
$-1, 0, 1, 0$	$-1, -1, 1, 0$

$0, 0, -1, -1$	$0, 1, 1, 1$
$1, 0, 1, 1$	$1, 1, 1, 1$

Does every PF-path converge to a NE?

No! RSP (Rapp)

Supermodular Games

$U_{ij} - U_{kj}$ is increasing in j for any $i > k$.

Then $x \mapsto BR(x)$ is increasing in x, w.r.t. stochastic dominance relation

$$x \leq y \Longleftrightarrow \sum_{i=k}^{n} x_i \leq \sum_{i=k}^{n} y_i \quad \forall k$$

This supermodularity in the static game is preserved in the perfect foresight dynamics: $V_i(\phi, t) - V_j(\phi, t)$ is increasing in ϕ for any $i > j$ and any t. $\Longrightarrow$ Comparison principle

d-absorbing $\Leftrightarrow$ absorbing

Theorem.[T]

Every generic supermodular 2 player game has exactly one d-absorbing strict Nash equilibrium, it is also globally accessible.

3×3 **symmetric supermodular games** [HS2]

$A = (a_{ij})_{i,j=1,2,3}$ 3 strict equilibria

select 2	if	$2 >> 1$ and $2 >> 3$
select 1	if	$1 >> 2 >> 3$ or
		$1 >> 2$, $3 >> 2$ and $q_1 > q_3$
select 3	if	$3 >> 2 >> 1$ or
		$1 >> 2$, $3 >> 2$ and $q_3 > q_1$

$2 >> 1$ means 2 risk–dominates 1 in absence of 3

$q_1 = \frac{a_{11}+a_{12}-a_{21}-a_{22}}{a_{21}+a_{23}-a_{11}-a_{13}}$ and $q_3 = \frac{a_{33}+a_{32}-a_{23}-a_{22}}{a_{21}+a_{23}-a_{31}-a_{33}}$

3 Person Unanimity Games

actions A_i, B_i $(i = 1, 2, 3)$

$$U_i(s) = \begin{cases} a_i & \text{if } s = A_1A_2A_3 \\ b_i & \text{if } s = B_1B_2B_3 \\ 0 & \text{otherwise,} \end{cases}$$

where $a_i, b_i > 0$.

$\mathbf{A}$ is said to have the higher Nash product if $\prod_i a_i > \prod_i b_i$.
(Harsanyi and Selten 1988)

PFD does not necessarily select the strict NE with the higher Nash product!

Example: $a_1 = a, a_2 = a_3 = 1$, $b_1 = b_2 = b_3 = 2$.

If $6 < a < 6 + 2\sqrt{6} = 10.9$ both $\mathbf{A}$ and $\mathbf{B}$ are globally accessible for small $\theta > 0$.

Open problem:

2 person zero-sum games:

Do all PF-paths converge to the set of equilibria?

Postdoc Position in Vienna

Dynamical systems and/or Stochastics applied to Game Theory or Math. Biology

starting March 2009

for 1 or 2 years

40.000 EUR