Lesson 7. The heat equation on Riemannian manifolds, by A. Savo

The aim of this short course is to introduce the fundamental facts concerning the heat equation on Riemannian manifolds. Rather than giving a systematic overview of the (old and new) results in the theory, we present a selected list of significant facts which are meant to motivate the research in this field. The main issue to be addressed here is the relation between heat diffusion and the geometry of the manifold ; this will be illustrated by examining geometric estimates of solutions (valid for all times) and exact asymptotics for small times.

- The heat kernel of Euclidean space. The heat kernel of a closed Riemannian manifold and its asymptotic expansion for small times. Heat kernel representation of a solution.

- The trace of the heat kernel and its relation with the spectrum of the Laplace operator. Isospectrality. Pointwise estimates. The maximum principle. Isoperimetric inequalities for the trace of the heat kernel. Applications to eigenvalue estimates.

- The Dirichlet heat kernel of a Riemannian manifold with boundary. Integral estimates. The heat kernel of a non-compact Riemannian manifold. Heat content.

- Heat content asymptotics. Overdetermined problems for the heat equation : stationary isothermal hypersurfaces of domains in Euclidean space. Related open questions on Riemannian manifolds.