Evolution of Gaussian measures for the one dimensional Schrödinger equation

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Introduction

Motivation : Long time behaviour of the nonlinear Schrödinger equation

$$\begin{cases} i\partial_s U + \partial_y^2 U = |U|^{p-1}U, & (s,y) \in \mathbb{R} \times \mathbb{R}, \\ U(0) = U_0 \in L^2(\mathbb{R}). \end{cases} \qquad (NLS_p)$$

- Almost sure global existence results ($p > 1$)
- Almost sure scattering results ($p > 3$)
- Evolution of Gaussian measures

On invariant measures

Definition 1

Consider a space X and one parameter group $(\Phi(t,.))_{t\in\mathbb{R}}$ with

$$\Phi(t,.) : X \longrightarrow X.$$

A measure μ defined in the space X is called invariant with respect to $(\Phi(t,.))_{t\in\mathbb{R}}$ if for any $\mu-$measurable set A

$$\mu\big(\Phi(t,A)\big) = \mu(A), \quad t \in \mathbb{R}.$$

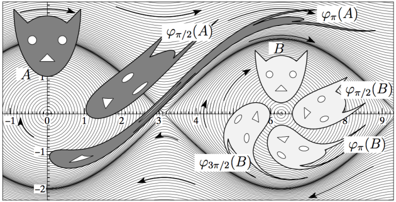

Figure: Conservation of the area for the pendulum.

On invariant measures: the Poincaré theorem

Theorem 2 (Poincaré)

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\Phi(t, .) : X \longrightarrow X$ be a map which preserves the probability measure μ.

(i) *Let $A \in \mathcal{B}$ be such that $\mu(A) > 0$, then there exists $k \geq 1$ such that*

$$\mu\big(A \cap \Phi(k, A)\big) > 0.$$

(ii) *Let $B \in \mathcal{B}$ be such that $\mu(B) > 0$, then for μ-almost all $x \in B$, the orbit $\big(\Phi(n, x)\big)_{n \in \mathbb{N}}$ enters infinitely many times in B.*

On invariant measures: the Liouville theorem

Consider the **ordinary differential equation**

$$\begin{cases} \dot{x}(t) = \dfrac{dx}{dt}(t) = F(x(t)), \\ x(0) = x_0. \end{cases}$$

We denote by $\Phi(t, \cdot)$ the flowmap of this system.

Theorem 3 (Liouville)

The flowmap $\Phi(t, .)$ preserves the measure $g dx$ if and only if

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \big(g F_k \big) = 0.$$

The Schrödinger equation on compact manifolds

- Let M a compact manifold. There exists a Hilbert basis $(h_n)_{n\geq 0}$ of $L^2(M)$, composed of eigenfunctions of Δ_M and we write

$$-\Delta_M h_n = \lambda_n^2 h_n \quad \text{for all} \quad n \geq 0.$$

- Consider a probability space $(\Omega, \mathcal{F}, \mathrm{p})$ and let $(g_n)_{n\geq 0}$ be a sequence of independent **complex standard Gaussian variables** $\mathcal{N}_{\mathbb{C}}(0,1)$.

$$g_n = \frac{1}{\sqrt{2}}(g_{1,n} + i g_{2,n}), \qquad g_{1,n}, g_{2,n} \in \mathcal{N}_{\mathbb{R}}(0,1).$$

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- Let $(\alpha_n)_{n\geq 0}$ and define the probability measure μ via the map

$$\omega \quad \longmapsto \quad \gamma(\omega) = \sum_{n=0}^{+\infty} \alpha_n g_n(\omega) h_n, \qquad \mu = \mathrm{p} \circ \gamma^{-1},$$

defined by : for all measurable set A,

$$\mu(A) = \mathrm{p}\big(\omega \in \Omega : \gamma(\omega) \in A\big).$$

The Schrödinger equation on compact manifolds

Proposition 4

The measure μ is invariant under the flow of the equation

$$i\partial_s U + \Delta_M U = 0, \quad (s, y) \in \mathbb{R} \times M.$$

In the nonlinear case, it is then natural to look for :

- invariant measures : **invariant Gibbs measures.**
- quasi-invariant measures.

The Schrödinger equation on compact manifolds

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- quasi-invariant measures.

Proof : For all $s \in \mathbb{R}$, the random variable

$$e^{is\Delta_M}\gamma(\omega) = \sum_{n=0}^{+\infty} \alpha_n e^{-is\lambda_n^2} g_n(\omega) h_n$$

has the same distribution as γ since for all $s \in \mathbb{R}$

$$e^{-is\lambda_n^2} g_n(\omega) \sim \mathcal{N}_{\mathbb{C}}(0, 1).$$

The non compact case

Proposition 5

Let $\sigma \in \mathbb{R}$ and consider a probability measure μ on $H^\sigma(\mathbb{R})$. Assume that μ is invariant under the flow Σ_{lin} of equation

$$\begin{cases} i\partial_s U + \partial_y^2 U = 0, & (s, y) \in \mathbb{R} \times \mathbb{R}, \\ U(0, \cdot) = U_0. \end{cases}$$

Then $\mu = \delta_0$.

- Same result for the equation

$$i\partial_s U + \partial_y^2 U = |U|^{p-1} U.$$

The non compact case

Proof :

- Let $\sigma \in \mathbb{R}$ and consider μ an **invariant probability measure** on $H^\sigma(\mathbb{R})$. Let $\chi \in C_0^\infty(\mathbb{R})$. By invariance of the measure,

$$\int_{H^\sigma(\mathbb{R})} \frac{\|\chi u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} d\mu(u) = \int_{H^\sigma(\mathbb{R})} \frac{\|\chi \Sigma_{lin}(t) u\|_{H^\sigma}}{1+\|\Sigma_{lin}(t) u\|_{H^\sigma}} d\mu(u),$$

and by **unitarity of the linear flow** in H^σ, we get

$$= \int_{H^\sigma(\mathbb{R})} \frac{\|\chi \Sigma_{lin}(t) u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} d\mu(u). \tag{1}$$

- Assume that the r.h.s. of (1) tends to 0 when $t \to +\infty$. This implies that $\|\chi u\|_{H^\sigma} = 0$, $\mu-$a.s., and thus $\mu = \delta_0$ since χ is arbitrary.

The non compact case

- By continuity of the product by χ in H^σ and unitarity of the linear flow in H^σ, we have

$$\frac{\|\chi\Sigma_{lin}(t)u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} \leq C\frac{\|\Sigma_{lin}(t)u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} = C\frac{\|u\|_{H^\sigma}}{1+\|u\|_{H^\sigma}} \leq C.$$

- If $v \in \mathcal{C}_0^\infty(\mathbb{R})$ is smooth, by the Leibniz rule and **dispersion**

$$\|\chi\Sigma_{lin}(t)v\|_{H^\sigma} \leq \|\chi\|_{W^{\sigma,4}}\|\Sigma_{lin}(t)v\|_{W^{\sigma,4}} \leq Ct^{-1/4}\|v\|_{W^{\sigma,4/3}} \longrightarrow 0,$$

when $t \longrightarrow +\infty$.

- Conclusion with an approximation argument.

Functional analysis

- Define the harmonic oscillator

$$H = -\partial_x^2 + x^2 .$$

- There exists a Hilbert basis $(e_n)_{n\geq 0}$ of $L^2(\mathbb{R})$, composed of eigenfunctions of H and we write

$$He_n = \lambda_n^2 e_n = (2n+1)e_n \quad \text{for all} \quad n \geq 0.$$

- We define the harmonic Sobolev space $\mathcal{W}^{\sigma,p}$ by the norm ($\sigma > 0$)

$$\|u\|_{\mathcal{W}^{\sigma,p}} = \|H^{\sigma/2}u\|_{L^p} \equiv \|(-\Delta)^{\sigma/2}u\|_{L^p} + \|\langle x\rangle^{\sigma}u\|_{L^p}.$$

(Yajima-Zhang)

The Gaussian measure: definition

- Let $\varepsilon > 0$, we define the probability **Gaussian measure** μ_0 on $\mathcal{H}^{-\varepsilon}(\mathbb{R})$ as the distribution of the random variable γ

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathcal{H}^{-\varepsilon}(\mathbb{R}) \\ \omega & \longmapsto & \gamma(\omega) = \displaystyle\sum_{n=0}^{+\infty} \frac{1}{\lambda_n} g_n(\omega) e_n, \end{array} \qquad \mu_0 = \mathrm{p} \circ \gamma^{-1}.$$

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- μ_0 is the **Gibbs measure** of the equation $i\partial_t u - Hu = 0$.

- We denote by $X^0(\mathbb{R}) = \bigcap_{\varepsilon>0} \mathcal{H}^{-\varepsilon}(\mathbb{R})$. Thus $L^2(\mathbb{R}) \subset X^0(\mathbb{R}) \subset \mathcal{H}^{-\varepsilon}(\mathbb{R})$.
- The measure μ_0 satisfies $\mu_0(L^2(\mathbb{R})) = 0$ and $\mu_0\big(X^0(\mathbb{R})\big) = 1$.

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- μ_0 is the **Gibbs measure** of the equation $i\partial_t u - Hu = 0$.
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- The measure μ_0 satisfies $\mu_0(L^2(\mathbb{R})) = 0$ and $\mu_0\big(X^0(\mathbb{R})\big) = 1$.
- The support of μ_0 is actually smoother

$$\mu_0\big(\big\{u_0 \in X^0(\mathbb{R}) : \|e^{-itH} u_0\|_{L^\infty((-\pi,\pi);\mathcal{W}^{1/6-,\infty})} \geq R\big\}\big) \leq C e^{-cR^2}.$$

Equivalence of Gaussian measures

- We say that $\mu \ll \nu$ if $\nu(A) = 0 \implies \mu(A) = 0$.

- Consider $\alpha_n, \beta_n > 0$ and the measures $\mu = \mathrm{p} \circ \gamma^{-1}$ and $\nu = \mathrm{p} \circ \psi^{-1}$ with

$$\gamma(\omega) = \sum_{n=0}^{+\infty} \frac{1}{\alpha_n} g_n(\omega) e_n, \qquad \psi(\omega) = \sum_{n=0}^{+\infty} \frac{1}{\beta_n} g_n(\omega) e_n.$$

Then the measures μ and ν are **absolutely continuous with respect to each other** (**same zero measure sets**) if and only if

$$\sum_{n=0}^{+\infty} \Big(\frac{\alpha_n}{\beta_n} - 1\Big)^2 < +\infty \qquad \text{(Kakutani)}.$$

- If the measures μ and ν are **not** absolutely continuous with respect to each other, they are **mutually singular** (Hajek-Feldman theorem).

The global existence result

Consider

$$\begin{cases} i\partial_s U + \partial_y^2 U = |U|^{p-1}U, & (s,y) \in \mathbb{R} \times \mathbb{R}, \\ U(0) = U_0 \in X^0(\mathbb{R}). \end{cases} \qquad (NLS_p)$$

Theorem 6

Let $p > 1$.

- *For μ_0-almost every initial data $U_0 \in X^0(\mathbb{R})$, there exists a unique, global in time, solution $U = \Psi(s,0)U_0$ to (NLS_p).*

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- *The measures $\Psi(s,0)_\# \mu_0$ and $\Psi_{lin}(s,0)_\# \mu_0$ are equivalent:*

$$\Psi_{lin}(s,0)_\# \mu_0 \ll \Psi(s,0)_\# \mu_0 \ll \Psi_{lin}(s,0)_\# \mu_0.$$

- In the compact case: $\Psi_{lin}(s,0)_\# \mu_0 = \mu_0$.

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$$\Psi_{lin}(s,0)_{\#}\mu_0 \ll \Psi(s,0)_{\#}\mu_0 \ll \Psi_{lin}(s,0)_{\#}\mu_0.$$

- *For all $s' \neq s$, the measures $\Psi(s,0)_{\#}\mu_0$ and $\Psi(s',0)_{\#}\mu_0$ are mutually singular.*

- In the compact case: $\Psi_{lin}(s,0)_{\#}\mu_0 = \mu_0$.

The scattering result

Theorem 7

▶ **Assume that** $p > 1$. *Then for* μ_0*−almost every initial data* $U_0 \in X^0(\mathbb{R})$, *there exists a constant* $C > 0$ *such that for all* $s \in \mathbb{R}$

$$\|\Psi(s,0)U_0\|_{L^{p+1}(\mathbb{R})} \leq \begin{cases} C\frac{(1+\log\langle s\rangle)^{1/(p+1)}}{\langle s\rangle^{\frac{1}{2}-\frac{1}{p+1}}} & \text{if } 1 < p < 5 \\ \frac{C}{\langle s\rangle^{\frac{1}{2}-\frac{1}{p+1}}} & \text{if } p \geq 5. \end{cases}$$

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- **Assume now that** $p > 3$. *Then there exist* $\eta > 0$ *and* $W_\pm \in L^2(\mathbb{R})$ *such that for all* $s \in \mathbb{R}$

$$\|\Psi(s,0)U_0 - e^{is\partial_y^2}(U_0 + W_\pm)\|_{L^2(\mathbb{R})} \leq C\langle s\rangle^{-\eta}.$$

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$$\|\Psi(s,0)U_0 - e^{is\partial_y^2}(U_0 + W_\pm)\|_{L^2(\mathbb{R})} \leq C\langle s\rangle^{-\eta}.$$

- For all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ we have the dispersion bound

$$\|e^{is\partial_y^2}\varphi\|_{L^{p+1}(\mathbb{R})} \leq \frac{C}{|s|^{\frac{1}{2}-\frac{1}{p+1}}}\|\varphi\|_{L^{(p+1)'}(\mathbb{R})}, \qquad s \neq 0,$$

therefore, the power decay in s is optimal.

Some references

Gross-Pitaevskii equation with random initial conditions and Gibbs measures

- Thomann 2009
- Burq–Thomann–Tzvetkov 2013
- Deng 2012
- Poiret–Robert–Thomann 2014
- Burq-Thomann 2023
- Latocca 2022
- Robert-Seong-Tolomeo-Wang 2023
- Dinh-Rougerie 2023
- Dinh-Rougerie-Tolomeo-Wang 2023

Gross-Pitaevskii equation with noise

- de Bouard–Debussche–Fukuizumi 2018, 2021 & 2022

Some references

Deterministic scattering results for NLS

- Barab 1984 $\longrightarrow$ never scattering when $p \leq 3$
- Tsutumi–Yajima 1984 $\longrightarrow$ scattering in L^2 with H^1 data
- Nakanishi 1999 $\longrightarrow$ scattering in H^σ
- Dodson 2016 $\longrightarrow$ scattering in L^2 when $p = 5$

Probabilistic scattering results for NLS

- Burq–Thomann–Tzvetkov 2013 $\longrightarrow$ case $d = 1$ and $p \geq 5$
- Poiret–Robert–Thomann 2014 $\longrightarrow$ case $d \geq 2$ and $p \geq 3$
- Killip–Murphy–Visan 2019 $\longrightarrow$ case $d = 4$ in the radial setting
- Latocca 2022 $\longrightarrow$ case $d \geq 2$ in the radial setting

Some references

Quasi-invariance for NLS and dispersive PDEs

- Tzvetkov 2015
- Oh-Tsutsumi-Tzvetkov 2019
- Oh–Tzvetkov 2017 & 2020
- Planchon–Tzvetkov–Visciglia 2020
- Debussche-Tsutsumi 2021

The lens transform: compactification in time and space

$$\begin{cases} i\partial_s U + \partial_y^2 U = |U|^{p-1} U, \quad (s,y) \in \mathbb{R} \times \mathbb{R}, \\ U(0) = U_0. \end{cases} \qquad (NLS_p)$$

If $U(s,y)$ is a solution of the problem (NLS_p), then the function $u(t,x)$ defined for $|t| < \frac{\pi}{4}$ and $x \in \mathbb{R}$ by

$$u(t,x) = \mathscr{L}(U)(t,x) := \frac{1}{\cos^{\frac{1}{2}}(2t)} U\Big(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\Big) e^{-\frac{ix^2 \tan(2t)}{2}}$$

solves the problem

$$\begin{cases} i\partial_t u - Hu = \cos^{\frac{p-5}{2}}(2t) |u|^{p-1} u, \quad |t| < \frac{\pi}{4}, \, x \in \mathbb{R}, \\ u(0,\cdot) = U_0. \end{cases}$$

Introduction

We consider the equation

$$\begin{cases} i\partial_t u - Hu = \cos^{\frac{p-5}{2}}(2t)|u|^{p-1}u, \quad (t,x) \in (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}, \\ u(0) = u_0. \end{cases}$$

- The energy

$$\mathcal{E}(t, u(t)) = \frac{1}{2}\|\sqrt{H}\, u(t)\|^2_{L^2(\mathbb{R})} + \frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})},$$

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is **not conserved**.

- For $-\frac{\pi}{4} < t < \frac{\pi}{4}$, we define the measure

$$\begin{aligned} d\nu_t &= e^{-\mathcal{E}(t,u)} du \\ &= e^{-\frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})}} d\mu_0 \end{aligned}$$

which is **not invariant**.

Monotonicity of the measure ν_t

We define the measure

$$d\nu_t = e^{-\frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})}} d\mu_0, \qquad -\frac{\pi}{4} < t < \frac{\pi}{4}.$$

Proposition 8

For all $0 \leq |t| < \frac{\pi}{4}$

$$\nu_0(A) \leq \begin{cases} \Big[\nu_t\big(\Phi(t,0)A\big)\Big]^{\big(\cos(2t)\big)^{\frac{5-p}{2}}} & \textit{if } 1 \leq p \leq 5 \\ \nu_t\big(\Phi(t,0)A\big) & \textit{if } p \geq 5. \end{cases}$$

- Allows to extend the globalisation argument of Bourgain relying on invariant measures.
- See the link with the Radon-Nikodym theorem.

A quantitative Radon-Nikodym theorem

Proposition 8

Let μ, ν be two finite measures on a measurable space $(X, \mathcal{T})$. Assume that

$$\mu \ll \nu,$$

and more precisely

$$\exists\, 0 < \alpha \leq 1, \quad \exists\, C > 0, \quad \forall A \in \mathcal{T}, \quad \mu(A) \leq C\nu(A)^{\alpha}.$$

By the Radon-Nikodym theorem, there exists a $f \in L^1(d\nu)$ with $f \geq 0$, such that $d\mu = f d\nu$, and we write $f = \dfrac{d\mu}{d\nu}$.

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By the Radon-Nikodym theorem, there exists a $f \in L^1(d\nu)$ with $f \geq 0$, such that $d\mu = f d\nu$, and we write $f = \dfrac{d\mu}{d\nu}$.

(i) *The assertion (2) is satisfied with $0 < \alpha < 1$ iff $f \in L^p_w(d\nu) \cap L^1(d\nu)$ with $p = \frac{1}{1-\alpha}$. In other words, $f \in L^1(d\nu)$ and*

$$\nu(\{x : \ |f(x)| \geq \lambda\}) \leq C' \langle \lambda \rangle^{-p}, \qquad \forall\, \lambda > 0.$$

(ii) *The assertion (2) is satisfied with $\alpha = 1$ iff $f \in L^\infty(d\nu) \cap L^1(d\nu)$.*

Monotonicity of the measure ν_t : proof

Proof :

- Set

$$\mathcal{E}(t,u(t)) = \frac{1}{2}\|\sqrt{H}\,u(t)\|^2_{L^2(\mathbb{R})} + \frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}.$$

A direct computation shows that

$$\frac{d}{dt}\big(\mathcal{E}(t,u(t))\big) = \frac{(5-p)\sin(2t)\cos^{\frac{p-7}{2}}(2t)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R})}.$$

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- Set $F(t) = \nu_t\big(\Phi(t,0)A\big)$. Then

$$\frac{d}{dt}F(t) = (p-5)\tan(2t)\int_A \alpha\big(t,u(t)\big)e^{-\mathcal{E}(t,u(t))}du_0,$$

where $\alpha(t,u) = \frac{\cos^{\frac{p-5}{2}}(2t)}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})}$.

Monotonicity of the measure ν_t : proof

- By Hölder, for all $k \geq 1$

$$\frac{d}{dt}F(t) \leq (p-5)\tan(2t)\frac{k}{e}\left(F(t)\right)^{1-\frac{1}{k}}.$$

- Optimisation with $k = -\log\left(F(t)\right)$ yields

$$\frac{d}{dt}F(t) \leq -(p-5)\tan(2t)\log\left(F(t)\right)F(t).$$

- Integration of the differential inequality

The Bourgain argument revisited

$$\begin{cases} i\partial_t u - Hu = \cos^{\frac{p-5}{2}}(2t)|u|^{p-1}u, & (t,x) \in (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}, \\ u_{|t=t_0} = u_0 \in X^0(\mathbb{R}). \end{cases}$$

- There exists a flow Φ such that the time of existence τ on the ball

$$B_R = \{u \in X^0(\mathbb{R}) \ : \ \|u\| \leq R^{1/2}\},$$

is uniform and such that $\tau \sim R^{-\kappa}$ for some $\kappa > 0$.

- Moreover, for all $|t| \leq \tau$

$$\Phi(t,0)\big(B_R\big) \subset \{u \in X^0(\mathbb{R}) \ : \ \|u\| \leq (R+1)^{1/2}\}.$$

- We have the large deviation estimate $\mu_0(X^0(\mathbb{R})\backslash B_R) \leq Ce^{-cR}$.

The Bourgain argument revisited

- For $T \leq e^{cR/2}$ fixed, we define the **set of the good data**

$$\Sigma_R = \bigcap_{k=-[T/\tau]}^{[T/\tau]} \Phi(k\tau,0)^{-1}(B_R).$$

- By the **monotonicity** of $\nu_t\big(\Phi(t,0)A\big)$:

$$\begin{aligned}
\nu_0(X^0(\mathbb{R})\backslash\Sigma_R) &\leq \sum_{k=-[T/\tau]}^{[T/\tau]} \nu_0\Big(\Phi(k\tau,0)^{-1}\big(X^0(\mathbb{R})\backslash B_R\big)\Big) \\
&\leq \sum_{k=-[T/\tau]}^{[T/\tau]} \nu_{k\tau}\big(X^0(\mathbb{R})\backslash B_R\big) \\
&\leq \big(2[T/\tau]+1\big)\mu_0\big(X^0(\mathbb{R})\backslash B_R\big) \\
&\leq ce^{-cR/2}
\end{aligned}$$

which shows that Σ_R is a big set of $X^0(\mathbb{R})$ when $R \longrightarrow +\infty$.

The Bourgain argument revisited

- We deduce that for all $|t| \leq T$ and $u \in \Sigma_R$

$$\|\Phi(t,0)(u)\| \leq (R+1)^{1/2}.$$

In particular, for $|t| = T \sim e^{cR/2}$

$$\|\Phi(t,0)(u)\| \leq C(\ln|t|+1)^{1/2},$$