Flow approach to the gKPZ equation

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EQFT measures

Aim: make sense of the formal expression

$$\nu(\mathrm{d}\phi) \propto e^{-V(\phi)} \mathcal{g}(\mathrm{d}\phi)\,,$$

where $\mathcal{g} = \mathcal{N}\big(0, (1-\Delta)^{-1}\big)$.

- (strong) UV problem: $\int \phi^4 = \infty$ a.s.;
- IR problem: ϕ has no decay as $\Lambda \uparrow \mathbb{R}^d$;
- large field problem: the potentiel V needs to be bounded below;
- (week) UV problem: even in perturbation, *convergent* subamplitudes create new divergences in the IR (called renormalon).

Let $\mathcal{g}_\varepsilon = \mathcal{N}\big(0, e^{-\varepsilon(1-\Delta)}/(1-\Delta)\big)$, and find V_ε such that

$$\nu_\varepsilon(\mathrm{d}\phi) = \frac{1}{Z_\varepsilon} e^{-V_\varepsilon(\phi)} \mathcal{g}_\varepsilon(\mathrm{d}\phi)$$

has a weak limit as $\varepsilon \downarrow 0$.

For $\mu > \varepsilon$,

$$\phi = \phi_{<\mu} + \phi_{\geqslant\mu}\,,$$

where

$$\mathrm{Law}(\phi_{<\mu}) = \mathcal{N}\big(0, (e^{-\varepsilon(1-\Delta)} - e^{-\mu(1-\Delta)})/(1-\Delta)\big)\,,\ \text{and}\ \mathrm{Law}(\phi_{\geqslant\mu}) = g_\mu\,.$$

One is interested in observables F such that

$$F(\phi) = F(\phi_{\geqslant\mu})\,,\ \text{for some}\ \mu > 0\,.$$

They verify

$$\begin{aligned}\mathbb{E}_{\nu_\varepsilon}[F(\phi)] &= \frac{1}{Z_\varepsilon}\mathbb{E}_{g_\varepsilon}[F(\phi)e^{-V_\varepsilon(\phi)}] = \frac{1}{Z_\varepsilon}\mathbb{E}_{\geqslant\mu}\Big[F(\phi_{\geqslant\mu})\mathbb{E}_{<\mu}[e^{-V_\varepsilon(\phi_{<\mu}+\phi_{\geqslant\mu})}]\Big] \\ &= \frac{1}{Z_\varepsilon}\mathbb{E}_{\geqslant\mu}[F(\phi_{\geqslant\mu})e^{-V_{\varepsilon,\mu}(\phi_{\geqslant\mu})}]\,,\end{aligned}$$

where we set $V_{\varepsilon,\mu}(\phi_{\geqslant\mu}) := -\log\mathbb{E}_{<\mu}[e^{-V_\varepsilon(\phi_{<\mu}+\phi_{\geqslant\mu})}]$.

Polchinski's flow equation [Polchinski, Kopper,...]

Hope: for any $\mu > \varepsilon$, $V_{\varepsilon,\mu}$ can be controlled uniformly in ε, provided one made the correct choice of "initial condition" $V_{\varepsilon,\varepsilon} \equiv V_{\varepsilon,0} = V_\varepsilon$.
By Gaussian integration, setting $C_{\varepsilon,\mu} = \int_\varepsilon^\mu e^{-t(1-\Delta)}\mathrm{d}t$ and $\dot{C}_\mu = e^{-\mu(1-\Delta)}$, one has

$$
\begin{aligned}
e^{-V_{\varepsilon,\mu}} &= e^{\frac{1}{2}\langle\nabla_\phi,\nabla_\phi\rangle_{C_{\varepsilon,\mu}}}\left(e^{-V_\varepsilon}\right)\\
\Rightarrow \partial_\mu e^{-V_{\varepsilon,\mu}} &= \frac{1}{2}\langle\nabla_\phi,\nabla_\phi\rangle_{\dot{C}_\mu} e^{-V_{\varepsilon,\mu}}\\
\Rightarrow \partial_\mu V_{\varepsilon,\mu} &= \frac{1}{2}\langle\nabla_\phi,\nabla_\phi\rangle_{\dot{C}_\mu} V_{\varepsilon,\mu} - \frac{1}{2}\langle\nabla_\phi V_{\varepsilon,\mu},\nabla_\phi V_{\varepsilon,\mu}\rangle_{\dot{C}_\mu}\,.
\end{aligned}
$$

Case $V(\phi) = \lambda\phi^4$. Try an ansatz, and expand

$$V_{\varepsilon,\mu}(\phi) = \sum_{i\geqslant 1}\lambda^i \sum_{n\geqslant 0}\int V^{i,n}_{\varepsilon,\mu}(\mathrm{d}x_1,\cdots,\mathrm{d}x_n)\phi(x_1)\cdots\phi(x_n)\,.$$

In $d = 3$, one ends up with

$$V_{\varepsilon,0} = \lambda\phi^4 + (a\lambda\varepsilon^{-1} + b\lambda^2\log\varepsilon^{-1})\phi^2 + (c\lambda\varepsilon^{-2} + d\lambda^2\varepsilon^{-1} + e\lambda^3\log\varepsilon^{-1})\,.$$

The large field problem

- The UV problem is solved, in the sense that we identified V_ε;
- The week UV problem too, since we were working with an *IR cut-off*;
- This is not the case of the large field problem: since V_ε is *not bounded below* (uniformly in $\varepsilon > 0$). This corresponds to the fact that the formal series defining $V_{\varepsilon,\mu}$ is divergent.

Two main options to handle this large field problem:

- Rather perform a discreet renormalization group: good factors coming from high convergent graphs can tame the divergence of V_ε;
- Combine the Langevin dynamic

  $$(\partial_t + 1 - \Delta)\phi = -\nabla_\phi V(\phi) + \xi$$

  with some PDE techniques.

Discrete flow approach [Kupiainen 2014]

In 2014, Kupiainen introduced a framework to solve singular SPDEs, based on a discrete renormalization group idea.

He deals with the dynamical Φ^4_3 equation:

$$\begin{aligned}(\partial_t + 1 - \Delta)\phi_\varepsilon &= -\lambda\phi_\varepsilon^3 + \mathfrak{c}_\varepsilon\phi_\varepsilon + \xi_\varepsilon =: S_\varepsilon[\phi_\varepsilon] \\ \Rightarrow \phi_\varepsilon &= G\big(\mathbf{1}_{t>0}S_\varepsilon[\phi_\varepsilon] + \delta_{t=0}\otimes\phi_\varepsilon(0)\big)\,.\end{aligned}$$

For simplicity, assume that formally, we are in the stationary case

$$\phi_\varepsilon(0) = G\big(1_{t\leqslant 0}S_\varepsilon[\phi_\varepsilon]\big)(0)$$

so that $\phi_\varepsilon = G\big(S_\varepsilon[\phi_\varepsilon]\big)$.

Define the *effective field*

$$\phi_{\geqslant\mu} \equiv \phi_{\varepsilon,\mu} := G_\mu\big(S_\varepsilon[\phi_\varepsilon]\big)\,, \text{ where } G_\mu \text{ is cut-off at scale } \mu\,,$$

along with the *effective force* by the relation

$$S_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}] = S_\varepsilon[\phi_\varepsilon]\,.$$

Discrete flow approach [Kupiainen 2014]

Note that a priori one does not have $\mathrm{D}V_{\varepsilon,\mu} = \mathbb{E}[S_{\varepsilon,\mu}]$.

On the other hand, recall that it holds

$$\mathbb{E}[e^{-V_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}]}] = \mathbb{E}[e^{-V_{\varepsilon}[\phi_{\varepsilon}]}] \, .$$

This motivates the definition of the effective force: by making $S_{\varepsilon,\mu}$ random, one has more room to require $S_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}] = \mathrm{cte}_{\varepsilon}$.

With a fixed point argument, Kupiainen constructed for a random $m \in \mathbb{N}$ a family

$$(S_{\varepsilon,2^{-n}})_{n \geqslant m}$$

starting from $S_{\varepsilon,2^{-\infty}} = S_{\varepsilon}$.

Involves tedious computations of stochastic objects.

The solution to the RG flow is local in scale, hence the solution to the equation is local in time.

The flow equation [Duch 21]

Recall that formally,

$$S_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}] = S_\varepsilon[\phi_\varepsilon]\,, \text{ and } \phi_{\varepsilon,\mu} = G_\mu\big(S_\varepsilon[\phi_\varepsilon]\big)\,.$$

Thus, one obtains a *flow equation*

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\mu} S_\varepsilon[\phi_\varepsilon] = \frac{\mathrm{d}}{\mathrm{d}\mu} S_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}] = \partial_\mu S_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}] + \mathrm{D}S_{\varepsilon,\mu}\partial_\mu\phi_{\varepsilon,\mu} \\ \Rightarrow 0 &= \partial_\mu S_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}] + \mathrm{D}S_{\varepsilon,\mu}\dot{G}_\mu S_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}]\,. \end{aligned}$$

Looks very much like the Polchinski flow equation.

Again, combined with an appropriate ansatz for $S_{\varepsilon,\mu}$

$$S_{\varepsilon,\mu}[\phi](x) = \sum_{i\geqslant 1}\lambda^i \sum_{n\geqslant 0}\int \xi^{i,n}_{\varepsilon,\mu}(x,\mathrm{d}y_1,\cdots,\mathrm{d}y_n)\phi(y_1)\cdots\phi(y_n)\,,$$

where the *force coefficients* $(\xi^{i,n}_{\varepsilon,\mu})$ are a collection of random variables polynomial in the noise, similar to the model of regularity structures.

Duch's cumulant flow equation [Duch 21]

The previous *flow equation* rewrites as a hierarchy of equations for the force coefficients

$$\partial_\mu \xi^{i,n}_{\varepsilon,\mu} = -\sum_j \sum_m (m+1)\xi^{i-j,m+1}_{\varepsilon,\mu} \dot{G}_\mu \xi^{j,n-m}_{\varepsilon,\mu} \,.$$

Duch made the following crucial remark: there exists a similar hierarchical system of equations for the cumulants

$$K^I_{\varepsilon,\mu} := \kappa_p\left(\xi^{i_1,n_1}_{\varepsilon,\mu}, \ldots, \xi^{i_p,n_p}_{\varepsilon,\mu}\right) \,,\ I = \big((i_1,n_1),\ldots,(i_p,n_p)\big)$$

of the force coefficients, reading

$$\partial_\mu K^I_{\varepsilon,\mu} = \sum_J C_{IJ} \dot{G}_\mu K^J_{\varepsilon,\mu} + \sum_{M,L} \tilde{C}_{IML} K^M_{\varepsilon,\mu} \dot{G}_\mu K^L_{\varepsilon,\mu} \,.$$

- It is therefore possible to construct all the cumulants (and therefore all the moments) of the force coefficients by induction, starting from the covariance of the noise;
- This analysis avoids much of the algebraic considerations present in regularity structures/higher paracontrolled calculus.

Beyond polynomial interactions

The generalized KPZ equation

$$(\partial_t - \Delta)\phi = g_{ij}(\phi)\partial_i\phi\partial_j\phi + h(\phi)\xi\,.$$

General case of semi-linear singular parabolic SPDE with non-polynomial interaction, including

- the Kardar–Parisi–Zhang equation,
- the multiplicative SHE, or parabolic Anderson model.

The equation is subcritical as long as the expected regularity of the solution is > 0 (i.e. $\xi \in \mathcal{C}^{-2+\alpha}$, $\alpha > 0$).

Falls outside the scope of the work by Duch, where

- the flow equation is implemented for a polynomial interaction;
- the solution theory is limited to the case where the regularity of the equation is negative.

Another motivation coming from the quantization of gauge theories:

$$\Phi_t(A_0^{g_0}) = \big(\Phi_t A_0\big)^{g_t} \text{ for } g^{-1}\partial_t g = -\mathrm{d}^*_{\Phi_t A_0}(g^{-1}\mathrm{d}g)\,.$$

Main result

For simplicity, we focus on gPAM:

$$(\partial_t - \Delta)\phi_\varepsilon = h(\phi_\varepsilon) + \mathfrak{c}_\varepsilon(h, \phi_\varepsilon, \partial_{\mathrm{x}}\phi_\varepsilon)\,.$$

Theorem 1 [Chandra, F.]

- Fix $\alpha \in (0 \vee (1/2 - n/4), 1]$, and let $\Gamma := \lfloor 2/\alpha - 1 \rfloor$;
- fix a function h in $C^{1+\Gamma+3N_1^{\Gamma+1}}(\mathbb{R})$;
- fix an initial condition $\phi_\varepsilon(0)$ in $C^{4N_1^{\Gamma+1}+1}(\mathbb{T}^n)$.

Then, there exists a random variable $0 < T \leqslant 1$ such that for any deterministic $\tilde{T} \in (0,1]$ the following holds on the event $\{\tilde{T} \leqslant T\}$: gPAM is well posed on $\mathcal{C}^{\alpha-}([0,\tilde{T}] \times \mathbb{T}^n)$ with solutions ϕ_ε which converges (in probability) to a limit ϕ in $\mathcal{C}^{\alpha-}([0,\tilde{T}] \times \mathbb{T}^n)$ as $\varepsilon \downarrow 0$.

Ansatz for the effective force

Basis for the flow equation spanned by multi-indices $a \in \mathbb{N}^{\mathbb{N}}$ (for gPAM) corresponding to some product of derivatives of h:

$$\Upsilon^a[\phi](y^a) := \prod_{i \in \operatorname{supp}(a)} \prod_{j \in [a_i]} h^{(i)}\big(\phi(y^a_{ij})\big) \,.$$

The multi-indices need to be *populated*, that is to say

$$\mathfrak{o}(a) := \sum_{i \geqslant 0} i a_i = 1 + \sum_{i \geqslant 0} a_i \,.$$

Subcriticality implies that it is sufficient to go to a certain order, after which all the force coefficients are *irrelevant*. In practice, for gPAM with $\alpha \in (2/3, 1]$, one has

$$\begin{aligned}
S_{\varepsilon,\mu}[\phi](x) &= \sum_{a : \mathfrak{o}(a) \leqslant 1} \langle \xi^a_{\varepsilon,\mu}, \Upsilon^a[\phi] \rangle(x) \\
&= \int \xi^{1,0,\cdots}_{\varepsilon,\mu}(x, \mathrm{d}y) h\big(\psi(y)\big) + \int \xi^{1,1,\cdots}_{\varepsilon,\mu}(x, \mathrm{d}y, \mathrm{d}z) h\big(\psi(y)\big) h'\big(\psi(z)\big) \\
&= \circ_\varepsilon h[\psi] + {}^{\circ}\!\!\diagup\!\!{}_{\circ}{}_{\varepsilon,\mu}(hh')[\psi] \,.
\end{aligned}$$

Renormalization

The force coefficients verify the flow equations

$$\partial_\mu \circ_{\varepsilon,\mu}(x,\mathrm{d}y) = 0 \;\Rightarrow\; \circ_{\varepsilon,\mu}(x,\mathrm{d}y) = \circ_\varepsilon(x,\mathrm{d}y) = \xi_\varepsilon(x)\delta_x(\mathrm{d}y) \text{ for all } \mu > 0\,,$$

and

$$\partial_\mu {\circ\!\!\diagup\!\!{}^{\circ}}_{\varepsilon,\mu}(x,\mathrm{d}y,\mathrm{d}z) = -\circ_\varepsilon(x,\mathrm{d}z)\int \dot{G}(z-w)\circ_\varepsilon(w,\mathrm{d}y)\mathrm{d}w\,.$$

The latter equation can not be solved forward in μ since $\|\partial_\mu {\circ\!\!\diagup\!\!{}^{\circ}}_{\varepsilon,\mu}\|_{L^\infty} \propto \mu^{-3+2\alpha}$, and is solved backward up to a (deterministic, diverging as $\varepsilon \downarrow 0$) value

$${\circ\!\!\diagup\!\!{}^{\circ}}_{\varepsilon,0}(x,\mathrm{d}y,\mathrm{d}z) = \mathfrak{c}_\varepsilon \delta_x(\mathrm{d}y)\delta_x(\mathrm{d}z)\,.$$

The counterterms are *local*.

Note that if we considered larger objects (of order larger than one), then they would vanish as $\mu \downarrow 0$.

The week UV problem in the flow approach

In the context of singular SPDEs, the new divergences caused by the terms convergent in the UV take the following from:

- they result in the fact that whenever $f \in \mathcal{C}^{\alpha>0}$ and $g \in \mathcal{C}^{\beta<0}$, fg is only $\mathcal{C}^{\beta}$ and not $\mathcal{C}^{\alpha+\beta}$.

Dealing with this difficulty is a the core of any solution theory to singular SPDEs:

- putting an IR cut-off on f, rather working with $f \succeq g$, suggests that the other piece $f \prec g$ should be added to a *paracontrolled ansatz*;
- recentering f, rather working with $\big(f - f(x)\big)g$ for some given base point x, suggests to add $f(x)g$ to the ansatz, and ultimately to view the solution as *modelled* by g, with coefficient $f(x)$;
- in the flow approach, an IR is directly implemented, in the sense that

$$
{}_{\circ}\!\diagup^{\!\circ}{}_{\varepsilon,\mu} = \circ_{\varepsilon}(G - G_{\mu})\circ_{\varepsilon} + \mathfrak{c}_{\varepsilon}\delta \,,
$$

  and the fluctuation propagator $G - G_{\mu} \propto \mu^{2}$ vanishes at short scales, so that indeed one has $\|{}_{\circ}\!\diagup^{\!\circ}{}_{\varepsilon,\mu}\|_{L^{\infty}} \lesssim \mu^{-2+2\alpha}$.

Construction of the solution 1: the non-stationary force

Going back to the original context, recall that we wanted to solve

$$\phi_\varepsilon = G\big(\mathbf{1}_{t>0} S_\varepsilon[\phi_\varepsilon] + \delta_{t=0} \otimes \phi_\varepsilon(0)\big)\,,\ S_\varepsilon[\phi_\varepsilon] = h[\phi_\varepsilon]\xi_\varepsilon + \mathfrak{c}_\varepsilon(hh')[\phi_\varepsilon]\,.$$

We constructed a *stationary* solution $S_{\varepsilon,\mu}$ to the flow equation truncated at order 1, with initial condition S_ε.

- To deal with parabolic problems, a good choice of UV cut-off is

  $$G_\mu(t,x) := \chi(t/\mu^2)e^{t\Delta}(x)\,,\ \text{supp}(\chi) = [1,\infty) \text{ and } \chi \restriction [2,\infty) = 1\,.$$

  Ensures that $\phi_{\varepsilon,\mu}$ is supported after time μ^2.
- Given the trajectory $(S_{\varepsilon,\mu})_{\mu>0}$, Duch showed that one can construct *without any additional renormalization* a trajectory $(F_{\varepsilon,\mu})_{\mu>0}$ solving the truncated flow equation with initial condition $\mathbf{1}_{t>0}S_\varepsilon$. Indeed, the stationary and non-stationary force coefficients agree after a time of order μ, and the small size of the remaining interval can be leveraged to complete the construction.
- The renormalization is therefore independent of time.

Construction of the solution 2: the remainder equation

To simplify, set $\phi_\varepsilon(0) = 0$, so that $\phi_\varepsilon = G\big(F_\varepsilon[\phi_\varepsilon]\big)$.

To compensate the fact that $F_{\varepsilon,\mu}$ only solves a truncated flow equation, we make the assumption that there exists a random time $T > 0$ along with a *remainder* $(R_{\varepsilon,\mu})_{\mu\in[0,\sqrt{T}]}$ such that for $\mu \in [0,\sqrt{T}]$,

$$F_\varepsilon[\phi_\varepsilon] = F_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}] + R_{\varepsilon,\mu}\,.$$

Again using $\frac{\mathrm{d}}{\mathrm{d}\mu}F_\varepsilon[\phi_\varepsilon] = 0$, one can derive the equation

$$\partial_\mu R_{\varepsilon,\mu} = -\mathrm{D}F_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}]\dot{G}_\mu R_{\varepsilon,\mu} - (\partial_\mu + \mathrm{D}F_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}]\dot{G}_\mu)F_{\varepsilon,\mu}[\phi_{\varepsilon,\mu}]\,,$$

that can be solved by a fixed point argument, but *locally in scale* up to a scale $\sqrt{T}$ (this is where the large field problem comes back into play).

On the other side, the solution reads

$$\begin{aligned}\phi_\varepsilon = G\big(F_\varepsilon[\phi_\varepsilon]\big) &= G\big(F_{\varepsilon,\sqrt{T}}[\phi_{\varepsilon,\sqrt{T}}] + R_{\varepsilon,\sqrt{T}}\big)\\ &= G\big(F_{\varepsilon,\sqrt{T}}[0] + R_{\varepsilon,\sqrt{T}}\big) \text{ on } [0,T]\,,\end{aligned}$$

and is therefore constructed.

Construction of the solution 3: the initial value problem

Let us go back to the initial value problem. $\delta_{t=0} \otimes \phi_\varepsilon(0)$ being too rough a forcing, one rather shifts by the harmonic completion of the initial condition

$$H := G\big(\delta_{t=0} \otimes \phi_\varepsilon(0)\big) = e^{t\Delta}\big(\phi_\varepsilon(0)\big) \,,$$

looking at $\psi_\varepsilon := \phi_\varepsilon - H$ which solves

$$\psi_\varepsilon = G\big(\mathbf{1}_{t>0} S_\varepsilon[\psi_\varepsilon + H]\big) = G\big(\mathbf{1}_{t>0} S_\varepsilon[\psi_\varepsilon + H]\big) = G\big(\mathbf{1}_{t>0}(S_\varepsilon[\psi_\varepsilon + H] - \delta_{t=0} \otimes \phi_\varepsilon(0))\big) \,.$$

In the polynomial case, Duch noted that given the trajectory $(S_{\varepsilon,\mu})_{\mu>0}$ solving the (possibly truncated) flow equation with initial condition S_ε,

$$\tilde{S}_{\varepsilon,\mu}[\psi] := S_{\varepsilon,\mu}[\psi + H_\mu] - \delta_{t=0} \otimes \phi_\varepsilon(0) \,,\; H_\mu = G_\mu\big(\delta_{t=0} \otimes \phi_\varepsilon(0)\big)$$

is a solution too, with initial condition $S_\varepsilon[\psi_\varepsilon + H] - \delta_{t=0} \otimes \phi_\varepsilon(0)$.

When the solution is of *negative* regularity say $\varsigma \leqslant 0$, $\tilde{S}_{\varepsilon,\mu}$ can be easily constructed, since H_μ shares the behavior of $(G - G_\mu)\circ_\varepsilon$, in the sense that one has

$$\|H_\mu\|_{L^\infty} \lesssim \mu^\varsigma \,.$$

In the case where the solution is of positive regularity $\alpha > 0$, things are different, since H_μ can definitely not behave like $(G - G_\mu) \circ_\varepsilon$.

Indeed, in the flow approach, being of positive regularity implies vanishing, while one has $H_\mu \to H$.

That would not be a problem if we had

$$\|H - H_\mu\|_{L^\infty} \lesssim \mu^\alpha \,.$$

However, the above estimate is wrong, since for $t \leqslant \mu^2$, $H_\mu(t) = 0$.

Actually, it only holds

$$\|t^{\alpha/2}(H - H_\mu)\|_{L^\infty} \lesssim \mu^\alpha \,.$$

The presence of many non-compactly supported kernels makes the flow equation with weights pretty involved.

Covariance blow-up [Hairer 2024]

As in other approaches, subcriticality is not the only limitation on the value of α.

Since $\mathord{\circ\!\!-\!\!\circ}_{\varepsilon,\mu} \propto \mu^{-2+2\alpha}$, one has

$$\mathrm{Cov}(\mathord{\circ\!\!-\!\!\circ}_{\varepsilon,\mu}, \mathord{\circ\!\!-\!\!\circ}_{\varepsilon,\mu}) \propto \mu^{-4+4\alpha} \in L^1 \Leftrightarrow -4 + 4\alpha + d + 2 > 0\,.$$

In $d = 1$, we need $\alpha > 1/4$.

Hairer showed in a very similar context, and in the marginal case $\alpha = 1/4$, that $\mathord{\circ\!\!-\!\!\circ}_{\varepsilon}$ converges to a new noise $\mathord{\circ\!\!-\!\!\circ}$ independent from $\circ$.

Moreover, the solution to the original KPZ equation converges to the solution to the KPZ equation driven by $\mathord{\circ\!\!-\!\!\circ}$.

Is there a way to "renormalize" the covariance?

Thank you!